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A design of disturbance observer in standard H_∞ control framework

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SUMMARY

The disturbance observer (DOB)-based controller is widely used to estimate and suppress disturbance in motion control system. Because the low-pass filter (Q-filter) in DOB decides the performances of disturbance suppression, noise rejection, and robust stability against system uncertainties, design of Q-filter is the principal task in DOB construction. This paper presents a systematic scheme for Q-filter design based on H_∞ norm optimization. Cost function for optimization is proposed by considering performance and relative order condition of the Q-filter. The norm minimization problem is then transformed to a standard H_∞ control problem. Furthermore, the relationship between performance and frequency weighting functions is investigated based on which selection of weighting functions is presented. Simulation results validate the global optimality and systematicness of the proposed method. Copyright © 2014 John Wiley & Sons, Ltd.

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KEY WORDS: disturbance observer; robust motion control; H_∞ control

1. INTRODUCTION

Disturbance observer (DOB)-based control method was originally proposed by Ohnishi in 1987 [1], whose effectiveness has been shown in estimation and suppression of system uncertainties and unknown disturbances in motion control system [2–6]. Especially, it has been widely applied in high-performance robust motion control due to its high disturbance rejection ability and simple structure. To design a DOB-based controller, an inner DOB loop (inner loop) is added into the control loop (outer loop), resulting in a control system of 2 DOF. Because the performance of system highly depends on a low-pass filter, called Q-filter, design of the filter has been essential in controller design and analysis, which has attracted much attention in academic society [7–10].

The first-order low-pass filter has been commonly used as Q-filter due to its simplicity in design [11, 12], whose time constant is used to adjust cutoff frequency of the filter. Later, it was recognized that the Q-filter with higher order can provide better performances of rejecting disturbance and noise. Thus, Butterworth, Chebyshev, or binomial coefficient typed high-order low-pass filters have been commonly used [8, 13, 14]. However, such typical low-pass filters were originally developed with the purpose of signal filtering and reproduction, whereas the missions of Q-filter in DOB are to suppress disturbance, sensor noise, and guarantee robustness against system uncertainties in control system. This implies that it is essential to investigate a proper design scheme as well as its evaluation for DOB.

Recently, several methods for designing DOB with H_∞ robust control technique have been reported [15–17]. It was shown that H_∞ norms of sensitivity functions of the system with DOB can

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sufficiently reflect the performance of attenuating disturbance and noise as well as the robust stability against system uncertainties [18]. To design a Q-filter that optimizes its norm cost function is, however, difficult because of restrictions such as relative order and internal model order. Thus, most of the design algorithms with H_∞ norm cost functions employed numerical computation methods [16, 17], such as LMI. The state-space results (DGKF hereafter) are widely accepted as an efficient and numerically sound way of computing H_∞ controllers [19]. Hence, in this paper, an optimization function is established based on the design requirements with DGKF method to achieve the optimal solution.

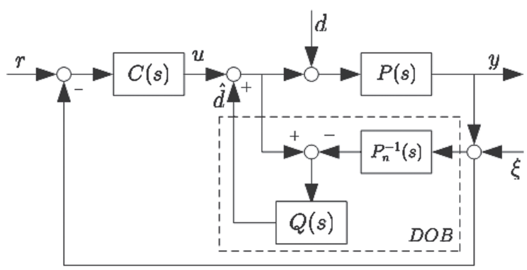
In this paper, a systematic strategy for DOB design is presented based on H_∞ norm optimization. All the performances of disturbance and noise attenuation are reflected in H_∞ norms of sensitivity functions as well as robust stability. Then, the relative order restriction of Q-filter is removed, hence, standard algorithm can be used to design the low-pass filter. Furthermore, the relationship between performance and frequency weighting functions is investigated based on which selection of weighting function is proposed.

The rest of this paper is organized as follows. A brief overview of DOB and the cost functions based on H_∞ norms of sensitivity functions is presented in Section 2. The loop shaping problem with relative order restriction for designing Q-filter is transformed into a standard H_∞ control problem and its solving algorithm is proposed in Section 3. Selections of the weighting functions are presented in Section 4. Sections 5 and 6 give the design example and application in hard disk drive system to illustrate the effectiveness of the proposed method, followed by conclusions in Section 7.

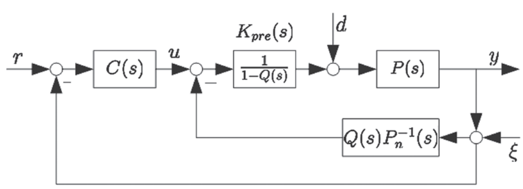
2. DOB AND ITS EVALUATION FUNCTION FOR DESIGN

Disturbance observer-based control system of 2-DOF structure is depicted in Figure 1, where $u, d, y, \hat{d}$, and ξ are command input, external disturbance, output, estimation of disturbance, and sensor noise, respectively. $P(s)$ and $P_n(s)$ represent the plant to be controlled and a nominal plant model, respectively. $Q(s)$ is a low-pass filter to be designed.

Disturbance observer considers the mismatch between plant and nominal model as equivalent disturbance acting on the nominal model. It estimates the equivalent disturbance combined with the external disturbance and feeds back the estimated disturbance for cancellation, as shown in Figure 1(a). A low-pass filter is introduced to avoid realization of non-proper inverse plant model and suppresses the sensor noise. Hence, design of DOB is reduced to design of low-pass filter $Q(s)$. DOB can fix the plant with parameters perturbation into nominal plant $P_n(s)$ by high gain feedback (see $K_{pre}(s)$ in Figure 1(b)).



(a) DOB based control system



(b) Equivalent form of DOB

Figure 1. Control structure of DOB.

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2.1. Overview

Behavior of the DOB loop can be analyzed by considering the transfer functions from u, d, and ξ to output y.

$$y = \frac{P_n P}{P_n + (P - P_n)Q} u - \frac{PQ}{P_n + (P - P_n)Q} \xi - \frac{P_n P(1-Q)}{P_n + (P - P_n)Q} d. \tag{1}$$

Assume that nominal model of the plant is correct (i.e., $P(s) = P_n(s)$) then (1) could be simplified as

$$y = P_n(s)u - Q(s)\xi - P_n(s)(1 - Q(s))d. \tag{2}$$

From (2), we can see that $Q(s)$ should go to zero in order to reject noise ξ, whereas $Q_C(s) := 1 - Q(s)$ should also be small in order to attenuate disturbance d, implying that $Q(s)$ should be 1. These two conditions are conflicting. In control applications, disturbance normally happens at low frequencies, whereas sensor noise and plant mismatch mainly take effects at high frequencies. This suggests that $Q(s)$ should be a low-pass filter and can be designed by weighted frequency minimization [7].

2.2. Robust stability

From Figure 1(a), it follows that for the plant with minimum phase property, in order to guarantee closed-loop stability in the condition of $P(s) = P_n(s)$, $Q(s)$ should be stable, which is called internal stability. The system should also hold the stability in the condition of system uncertainties, that is, robust stability. Suppose that the system uncertainties can be treated as the following multiplicative perturbation

$$P(s) = P_n(s)(I + \Delta(s)), \tag{3}$$

where $\Delta(s)$ is system uncertainties and assumed to be stable. The block diagram regarding the robust stability of inner loop is shown in Figure 2. DOB loop is robustly stable by small gain theorem if

$$\bar{\sigma}(\Delta(j\omega)Q(j\omega)) < 1, \forall\omega.$$

Assume that a stable $W_T(s)$ is an upper bound frequency function of $\Delta(s)$, then robust stability is satisfied if

$$\|W_T(s)Q(s)\|_\infty \leqslant 1. \tag{4}$$

This is used as one of the restrictions in the design of low-pass filter $Q(s)$.

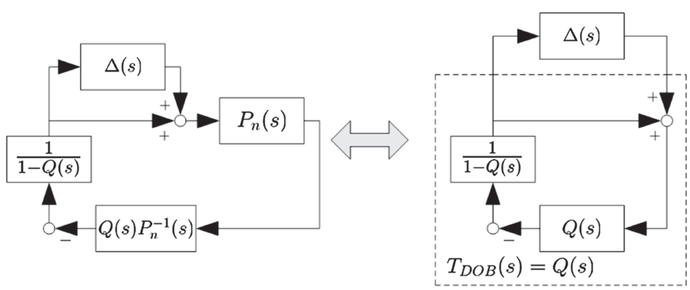

Figure 2. Robust stability of the inner loop.

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2.3. *Order and relative order conditions*

Disturbance observer includes an inverse nominal plant model $P_n^{-1}(s)$, which is unrealizable because it is non-proper. From Figure 1(b), it is clear that the relative order of $Q(s)$ should be larger than or at least equal to that of the nominal model $P_n(s)$, which is a critical restriction to the filter design procedure to enable practical implementation of DOB.

2.4. *Internal model order condition*

In order to attenuate a certain disturbance perfectly, the controller $K_{pre}(s)$ should include the model of the disturbance. Suppose that the disturbance to be rejected can be described as $d(t) = p_q t^q + p_{q-1}t^{q-1} + \cdots + p_0$. According to the internal model principle [20], it follows that the controller $K_{pre}(s)$ should include $q+1$ integrating actions. Thus, q is important to the design of the low-pass filter $Q(s)$.

2.5. *Sensitivity analysis*

The sensitivity function $S_{DOB}(s)$ and complementary sensitivity function $T_{DOB}(s)$ of inner loop are the transfer functions from equivalent disturbance at output terminal and control input u to output y, respectively. From Figure 1(b), we can easily derive [7, 18]

$$\begin{cases} S_{DOB} = \dfrac{P_n(1-Q)}{P_n + (P-P_n)Q}, \\ T_{DOB} = \dfrac{PQ}{P_n + (P-P_n)Q}. \end{cases} \tag{5}$$

We first analyze the nominal performance under condition $P(s) = P_n(s)$. This results in $T_{DOB}(s) = Q(s)$ and $S_{DOB}(s) = Q_C(s)$ [21]. The sensitivity and complementary sensitivity functions $S_{clo}(s)$ and $T_{clo}(s)$ of the whole closed-loop system in Figure 1 are the transfer functions from equivalent disturbance at output terminal and reference input r to output y, respectively, which are given as [16]

$$\begin{cases} S_{clo} = \dfrac{1-Q}{1+P_nC} = \dfrac{S_{DOB}}{1+P_nC}, \\ T_{clo} = \dfrac{P_nC + Q}{1+P_nC} = \dfrac{P_nC + T_{DOB}}{1+P_nC}. \end{cases} \tag{6}$$

From (5) and (6), it is shown that in order to achieve disturbance suppression, robustness against model uncertainties, and noise rejection performance, it is desirable to reduce $Q(s)$ and $Q_C(s)$ as much as possible. Because disturbance is normally at low frequency, whereas measurement noise is at high frequencies, $Q_C(s)$ should be minimal in low frequencies (control band), whereas $Q(s)$ should be minimal in high frequencies (model uncertainty and noise band).

2.6. *Cost function of DOB*

On the basis of aforementioned design restrictions and performances, we can now define the cost function of DOB by the H_∞ norm as

$$\max \gamma, \quad \min_{\substack{Q(s)\in\Omega_K \\ Q(s)\in RH_\infty}} \left\| \begin{bmatrix} \gamma W_C(s)(1-Q(s)) \\ W_Q(s)Q(s) \end{bmatrix} \right\|_\infty < 1, \tag{7}$$

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where

$$\Omega_k = \left\{ G(s) | G(s) = \frac{M(s)}{N(s)}, N(s) = \sum\nolimits_{i=0}^{n} a_i s^i, M(s) = \sum\nolimits_{j=0}^{m} b_j s^j \right. \\ \left. a_n \neq 0, b_m \neq 0, n - m = k \right\} \tag{8}$$

is the set of transfer functions that satisfies the relative order condition k. Weighting function $W_C(s)$ reflects the frequency band of disturbance to be rejected and its inverse represents the desired sensitivity at low frequencies. Internal model order condition is also included in the weighting factor as shown later. Similarly, $W_Q(s)$ reflects frequency band of sensor noise and the upper bound of model uncertainties while its inverse restricts the desired complementary sensitivity at high frequencies. The order of $Q(s)$ is determined by k as well as the order of $W_C(s)$ and $W_Q(s)$.

Equation (7) is a mixed weighting sensitivity problem in which the trade-off between two conflicting minimizations of $Q(s)$ and $1 - Q(s)$ is resolved by minimizing them in each frequency band determined by $W_Q(s)$ and $W_C(s)$. By maximizing γ, the optimal $Q(s)$ can be obtained, which is robust against model uncertainty and has good performance of disturbance and noise rejection. However, this H_∞ norm optimization problem is difficult to solve systematically because of restrictions such as relative order and internal model order, as shown in Section 4. In the next section, a method for solving problem (7) in the framework of standard H_∞ control problem is discussed.

3. Q-FILTER OPTIMIZATION

The optimization problem in (7) is complicated with relative and internal model order constraints. Standard H_∞ control theory provides generalized analysis and synthesis tools for a variety of control problems in which the design requirements are described by H_∞ norm constraint. Meanwhile, the algorithms such as DGKF method give simple and systematic solutions for the standard problem [19]. However, problem (7) cannot be solved in similar way due to extra restrictions unacceptable in standard scheme. This section presents a method of transforming the problem (7) to standard problem without order constraints.

3.1. Transforming into standard H_∞ problem

Define a scalar pseudo loop function $\tilde{L}(s)$ by $Q(s)$ as

$$\tilde{L}(s) := Q(s)(1 - Q(s))^{-1}. \tag{9}$$

Then the sensitivity and complementary sensitivity functions of DOB system are

$$Q(s) = \tilde{L}(s)(1 + \tilde{L}(s))^{-1}, Q_C(s) = 1 - Q(s) = \left(1 + \tilde{L}(s)\right)^{-1}, \tag{10}$$

where $\tilde{L}(s)$ can be regarded as the open-loop transfer function of a pseudo system (in fact, it is equivalent to the open-loop transfer function of the inner loop in Figure 1(b) under condition of model uncertainty). It is clear that $\tilde{L}(s) \in \Omega_k$ if and only if $Q(s) \in \Omega_k$, that is, the relative order of $\tilde{L}(s)$ is same as that of $Q(s)$.

Then (7) can be rewritten as

$$\max \gamma, \min_{\substack{\tilde{L}(s) \in \Omega_k \\ \tilde{L}(s) \in \Pi}} \left\| \begin{bmatrix} \gamma W_C(s) \left(1 + \tilde{L}(s)\right)^{-1} \\ W_Q(s) \tilde{L}(s) \left(I + \tilde{L}(s)\right)^{-1} \end{bmatrix} \right\|_\infty < 1, \tag{11}$$

where Π is the set of loop transfer functions, which make the corresponding closed-loop system stable. In this way, the problem (7) can be transformed into problem (11) perfectly where the optimal open-loop transfer function $\tilde{L}(s)$ should be solved and then $Q(s)$ is computed by (10). However,

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the problem still contains the restriction $\tilde{L}(s) \in \Omega_k$. Let $\tilde{L}(s)$ be factorized into pseudo plant $\tilde{P}(s)$ and pseudo controller $\tilde{K}(s)$ as follows:

$$\tilde{L}(s) = \tilde{P}(s)\tilde{K}(s). \tag{12}$$

Assumption 1
Virtual plant $\tilde{P}(s)$ satisfies the following conditions:

1. $\tilde{P}(s)$ does not have any unstable poles;
2. $\tilde{P}(s)$ does not have any zeros except infinite zeros;

We can simply select $\tilde{P}(s) = \alpha/(s+\beta)^k$ for arbitrary real number $\alpha > 0$ and $\beta > 0$. Note that the pseudo plant $\tilde{P}(s)$ is independent of plant $P(s)$ and only preserves the relative order of $P_n(s)$ so as to realize the relative order demand of loop transfer function to be designed. Furthermore, the way to factorize $\tilde{L}(s)$ is independent of $P(s)$ and the restriction on $\tilde{L}(s)$ is only $\tilde{L}(s) \in \Omega_k \cap \Pi$. Substitute (12) into (11), then we have the following optimization problem with respect to controller $\tilde{K}(s)$

$$\max \gamma, \min_{\tilde{K}(s)\in\Pi} \left\| \begin{bmatrix} \gamma W_C(s)\left(I + \tilde{P}(s)\tilde{K}(s)\right)^{-1} \\ W_Q(s)\tilde{P}(s)\tilde{K}(s)\left(I + \tilde{P}(s)\tilde{K}(s)\right)^{-1} \end{bmatrix} \right\|_\infty < 1, \tag{13}$$

Theorem 1
Assume that the nominal plant $\tilde{P}(s)$ of H_∞ norm optimization problem (13) satisfies Assumption 1. Let $\tilde{K}^*(s)$ be the optimal solution of (13). The optimal loop function $\tilde{L}^*(s) = \tilde{P}(s)\tilde{K}^*(s)$ is independent of the selection of $\tilde{P}(s)$ and uniquely determined.

Proof
Let Σ_p be a set of virtual plants that satisfies Assumption 1. Assume that $\tilde{P}(s) \in \Sigma_p$ is a virtual plant with which the corresponding optimal controller $\tilde{K}^*(s)$ makes the norm in problem (13) minimal. Then $\tilde{P}(s)$ has k stable poles $p_i^*(i = 1, \cdots, k)$. From loop shaping theory, it is clear that the optimal solution $\tilde{K}^*(s)$ of problem (13) includes k zeros equal to these poles as

$$\tilde{K}^*(s) = \tilde{K}_0(s)\prod_{i=1}^{k}\left(s + p_i^*\right). \tag{14}$$

Because $\tilde{L}^*(s) = \tilde{P}(s)\tilde{K}^*(s) = \tilde{K}_0(s)$ is an optimal open-loop function. For another virtual plant, we can get the similar conclusion and $\tilde{K}_0(s)$ is also an optimal open-loop function for $\tilde{P}_1(s)$. This means that selection of $\tilde{P}(s)$ satisfying Assumption 1 does not have any influence on minimizing the norm of problem (13), resulting in $\tilde{L}^*(s) = \tilde{K}_0(s)$. □

Lemma 1
Let $\tilde{K}(s)$ be an optimal solution of order n and relative order k for any $\tilde{P}(s) \in \Sigma_p$ in (13). Then transfer function $\tilde{L}^*(s) = \tilde{P}(s)\tilde{K}(s)$ of order n and relative order k is the optimal loop function that gives the smallest norm value of (11) among other transfer functions of the same order construction.

Lemma 2
For any $\tilde{P}(s) \in \Sigma_p$, two H_∞ norm optimization problems (11) and (13) are perfectly equivalent and

$$\tilde{L}^*(s) = \tilde{P}(s)\tilde{K}^*(s), \tag{15}$$

where $\tilde{L}^*(s)$ and $\tilde{K}^*(s)$ are the optimal solutions of problems (11) and (13), respectively.

Proofs of the Lemmas 1 and 2 are similar to that of Theorem 1. Equation (13) is a mixed weighting sensitivity problem without relative order restriction and can be solved by the framework of standard

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H_∞ control problem. Hence, it is easy to know that the H_∞ norm optimization problem (11) with relative order restriction can be transformed into standard H_∞ control problem (13) without order restriction, which can be systematically solved in a normal way. The order of $\tilde{K}^*(s)$ does not exceed one of the augmented plants of standard H_∞ problem that consists of weighting functions and virtual plant. The order of $\tilde{L}^*(s)$ is the same as that of $\tilde{K}^*(s)$ because all the poles of $\tilde{P}(s)$ are cancelled by zeros of $\tilde{K}^*(s)$. $\tilde{L}^*(s)$ satisfies $\tilde{L}^*(s) \in \Omega_k$ because $\tilde{K}^*(s)$ is proper, thus $Q^*(s) \in \Omega_k$.

3.2. *Design procedure of optimal Q-filter*

From the discussions shown earlier, Q-filter can be designed in the following procedures:

- Step 1. Determine the weighting functions $W_C(s)$ and $W_Q(s)$ that reflect the design specifications such as frequency response requirements and internal model order (as seen later).
- Step 2. Select an arbitrary $\tilde{P}(s) \in \Sigma_p$ that reflects relative order condition k of Q-filter.
- Step 3. According to the solving framework of standard H_∞ control problem, solve the optimal solution $\tilde{K}^*(s)$ of the norm optimization problem (13).
- Step 4. According to (15), obtain optimal loop function $\tilde{L}^*(s)$ and then compute the optimal Q-filter $Q(s)$ by (10).

In standard problem of Step 3, DGKF solution [19] based on two Ricatti equations can be effectively used. Note that $Q^*(s)$ and $\tilde{K}^*(s)$ have the same order, which depends on the virtual plant and order of the augmented plant.

4. INTERNAL MODEL ORDER AND SELECTION OF WEIGHTING FUNCTIONS

It is essential for a DOB system to attenuate specified disturbance perfectly. For example, if the system should perfectly suppress the disturbance $d(t)$ mentioned in Section 2.4, then q becomes an design specification that should be attained by Q-filter. From the internal model principle, it follows that the equivalent controller K_{pre} in Figure 1(b) should include $q+1$ integral factors in order to eliminate the effect of the disturbance on output. Thus, internal model order q should be taken into account. Here, we consider the problem of suppressing the disturbances in the form of temporal polynomial to be completely eliminated.

4.1. *Analysis*

Assume that the minimum realization of $Q(s)$ is as follows:

$$Q(s) = \frac{M(s)}{N(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{j=0}^{n} a_j s^j}, a_n \neq 0, b_m \neq 0, n > m, n-m = k, \tag{16}$$

where n and k are the order and relative order of $Q(s)$, respectively. $M(s)$ and $N(s)$ are coprime polynomials.

Theorem 2
Suppose that the nominal plant $P_n(s)$ is bounded on imaginary axis. DOB has given internal model order q if and only if $b_l = a_l (l = 0, 1, \cdots q, 0 \leqslant q \leqslant m)$.

Proof
$K_{pre}(s)$ in Figure 1(b) can be written as

$$K_{pre}(s) = \frac{1}{1-Q(s)} = \frac{N(s)}{N(s)-M(s)} = \frac{N(s)}{D(s)},$$

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$$D(s) = s^{q+1}\left[\sum\nolimits_{i=m+1}^{n} a_i s^{i-q-1} + \sum\nolimits_{j=q+1}^{m} (a_j - b_j)s^{j-q-1}\right] + \sum\nolimits_{l=0}^{q}(a_l - b_l)s^l,$$
$$N(s) = \sum\nolimits_{i=0}^{n} a_i s^i. \tag{17}$$

Suppose that

$$b_l = a_l, l = 0, 1, \cdots, q. \tag{18}$$

Then, it is clear that $K_{pre}(s)$ may have $q+1$ integrating factors at most. Moreover, cancellation of factors between numerator and denominator never appears in (17), because $Q(s) \in RH_\infty \Rightarrow a_i > 0 (i = 0, \cdots, n)$. It implies that integrating order of $K_{pre}(s)$ is not smaller than $q+1$. It means that $K_{pre}(s)$ has $q+1$ integrators. On the contrary, from (17), it is clear that if DOB has $q+1$ integrating factors, then (18) is satisfied. □

Lemma 3
The reachable maximum value of internal model order q is the same as the numerator's order n.

Proof of Lemma 3 is similar to that of Theorem 2. This lemma shows the important information of DOB that the higher numerator's order is, the higher reachable maximum internal model order will be. However, because there exists the relation of $n = m + k$, the order n should also get higher in order to keep the original relative order.

Theorem 3
The solution $Q^*(s)$ of the H_∞ norm optimization problem (7) satisfies the given internal model order demand q if weighting function $W_C(s)$ includes $q+1$ poles at the origin of the complex plane.

Proof
Let $Q^*(s)$ be the solution of problem (7), then $W_C(s)Q_C^*(s) \in RH_\infty$. Thus, if $W_C(s)$ includes $q+1$ poles at origin point, $Q_C(s) = 1 - Q^*(s)$ should include $q+1$ zeros at the origin so as to cancel all the poles at the origin of $W_C(s)$. However, we have

$$Q_C(s) = \frac{D(s)}{M(s)} = \frac{\sum_{i=0}^{n} a_i s^i - \sum_{j=0}^{m} b_j s^j}{\sum_{i=0}^{n} a_i s^i}. \tag{19}$$

When the numerator of $Q_C(s)$ has $q+1$ zeros at the origin, its coefficients of the lowest q terms are zero. Therefore, (18) is satisfied. □

From Theorem 3, it follows that only if the weighting function $W_C(s)$ is selected to have $q+1$ integrating factors as follows, then the internal model order specification q can be achieved as

$$\bar{W}_C(s) = \gamma W_C(s) = \gamma \frac{1}{s^{q+1}} \hat{W}_C(s). \tag{20}$$

However, if $W_C(s)$ has poles on the imaginary axis, the augmented plant of standard problem could not satisfy the preconditions for solution, because of the invariant zeros (corresponding to uncontrollable or undetectable states) on the imaginary axis. In order to avoid this problem in finding solution, a sufficiently small parameter $\lambda > 0$ can be introduced. Then, we should use the following weighting function instead of (20)

$$\bar{W}_C(s) = \gamma W_C(s) = \gamma \frac{1}{(s+\lambda)^{q+1}} \hat{W}_C(s). \tag{21}$$

The resultant controller includes $q+1$ poles $s_i = -\lambda (i = 0, 1, \cdots, q)$, that is,

$$\tilde{K}^*(s, \lambda) = \tilde{K}^*_{\lambda,0}(s) \frac{1}{(s+\lambda)^{q+1}}. \tag{22}$$

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When $\lambda \to 0$, the solution of standard problem goes to real solution

$$\tilde{K}^*(s, 0^+) = \lim_{\lambda \to 0^+} \tilde{K}^*(s, \lambda) = \tilde{K}_0^*(s)\frac{1}{s^{q+1}}.$$

Practically, we can use a small real number $\lambda > 0$ to get a sufficiently accurate solution as

$$\tilde{K}_\lambda^*(s) = \tilde{K}_{\lambda,0}^*(s)\frac{1}{s^{q+1}}. \tag{23}$$

$\tilde{K}^*(s, \lambda)$ certainly makes the corresponding closed loop internally stable with $\tilde{P}(s)$ because it is a real solution of the problem.

Theorem 4
There exists a sufficiently small real number $\lambda > 0$ such that the closed-loop system is stable or equivalently, $Q^*(s)$ is stable for the λ.

Proof
The optimal Q-filter for weighting function (21) has the form as (22). Then, we have

$$Q^*(s, \lambda) = \frac{\tilde{L}^*(s, \lambda)}{1 + \tilde{L}^*(s, \lambda)} = \frac{\tilde{K}_{\lambda,0}^*\tilde{P}(s)/(s+\lambda)^{q+1}}{1 + \tilde{K}_{\lambda,0}^*\tilde{P}(s)/(s+\lambda)^{q+1}}. \tag{24}$$

For simplicity, define

$$\tilde{N}(s)/\tilde{D}(s) := \tilde{K}_{\lambda,0}^*(s)\tilde{P}(s), \tag{25}$$

where $\tilde{N}(s)$ and $\tilde{D}(s)$ are coprime. Then we have

$$Q^*(s, \lambda) = \frac{\tilde{N}(s)}{\tilde{D}(s)(s+\lambda)^{q+1} + \tilde{N}(s)} = \frac{\tilde{N}(s)}{\tilde{D}_C^*(s, \lambda)},$$

where $D_C^*(s, \lambda) = \tilde{D}(s)(s+\lambda)^{q+1} + \tilde{N}(s)$. D_C^* is stable because the optimal solution $\tilde{K}^*(s, \lambda)$ makes the closed-loop system stable. If we employ $\tilde{K}_\lambda^*$ instead of $\tilde{K}^*(s, \lambda)$, then the corresponding Q-filter could be

$$Q(s) = Q^*(s, 0) = \frac{\tilde{N}(s)}{\tilde{D}(s)s^{q+1} + \tilde{N}(s)} = \frac{\tilde{N}(s)}{\tilde{D}_C(s)},$$

where $\tilde{D}_C(s) = \tilde{D}(s)s^{q+1} + \tilde{N}(s)$. By Kharitonov theorem, $\tilde{D}_C(s)$ is stable if the coefficient's values of the error polynomial $\tilde{D}_C(s) - \tilde{D}_C^*(s, \lambda)$ are restricted to a certain region of coefficient hyperplane or equivalently, $\tilde{D}_C(j\omega) \neq 0$, $\forall\omega$, that is,

$$\tilde{D}_C(j\omega) = \tilde{D}_C^*(j\omega, \lambda) + \left[\tilde{D}_C(j\omega) - \tilde{D}_C^*(j\omega)\right] = \tilde{D}_C^*(j\omega, \lambda) + \delta(j\omega) \neq 0, \tag{26}$$

where $\delta(s) = \tilde{D}_C(s) - \tilde{D}_C^* = [s^q - (s+\lambda)^q]\tilde{D}(s)$ is the error. Because $\tilde{D}_C^*(s, \lambda)$ is stable, $\tilde{D}_C^*(j\omega, \lambda) \neq 0$ for $\forall\omega$. Divide (26) by $\tilde{D}_C^*(s, \lambda)$, then we have

$$1 + \frac{\delta(j\omega)}{\tilde{D}_C^*(j\omega, \lambda)} \neq 0, \forall\omega. \tag{27}$$

$\tilde{D}_C(s)$ is stable if and only if the locus of $\delta(j\omega)/\tilde{D}_C^*(j\omega, \lambda)$, $\forall\omega$, does not surround the Nyqist point $(-1, j0)$ or equivalently,

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$$\sup_{\omega}\left|\frac{\delta(j\omega)}{\tilde{D}_C^*(j\omega,\lambda)}\right| = \left\|\frac{(s^q-(s+\lambda)^q)\tilde{D}(s)}{\tilde{D}_C^*(s,\lambda)}\right\|_\infty < 1. \tag{28}$$

It is clear that there exists $0 < \lambda < \lambda_{\max}$ such that (28) is satisfied. □

4.2. *Selection guide of weighting functions*

From the analysis stated earlier, we can now get the selection guide of the weighting functions $W_C(s)$ and $W_Q(s)$ in Step 1 of Q-filter design procedure in Section 3. Suppose that the order n, internal model order q, and relative order k of the Q-filter to be designed are, respectively, prescribed.

Selection of $W_Q(s)$. $W_Q(s)$ restricts $Q(s)$ on high frequencies and its inverse relative order should be the same as the relative order of $\tilde{P}(s)$ so as to satisfy precondition of standard H_∞ control problem. The coefficients of $W_Q(s)$ should be selected that considers frequency response for both robust stability and noise rejection performance.

Selection of $W_C(s)$. The order n_k of designed $\tilde{K}^*(s)$ is determined by the order $n_W + k$ of augmented plant of the standard problem, where n_W is the order of $W_C(s)$. In suboptimal problem, n_K is equal to $n_W + k$ in the case that $W_Q(s)$ does not have its modal. However, one modal of the controller at least gradually vanishes as the suboptimal controller approaches optimal one by γ-maximization, resulting in $n_K = n_W + k - 1$. Therefore, in order to attain the order demand $n_K = n$ of $Q(s)$, n_W should be selected as $n_W = n - k + 1 = m + 1$. The highest internal model order can be realized with the condition of $q_{\max} = n_W - 1$, then $W_C(s)$ should include $n_I = q_{\max} + 1 = n_W$ integrating factors. In this case, $\hat{W}_C(s)$ in (20) should be selected to be a polynomial, whose order is allowed to be n_W at most. The detailed coefficients of this polynomial should be selected considering frequency response of disturbances at low frequencies.

4.3. *Implementation of Q-filter*

When $\tilde{K}(s)$ is solved by Step 3 of Q-filter design procedure described in Section 3, the integrating factors should be taken into account in order to attain internal model order demand. Select $W_C(s)$ to include n_I integrating factors as previously mentioned and then use a positive but sufficiently small real number $\lambda > 0$ to reconstruct $W_C(s)$ by (21) so as to satisfy preconditions of solving algorithm. After solving standard H_∞-control problem, the resultant solution $\tilde{K}^*(s,\lambda)$ must be replaced by $\tilde{K}_\lambda^*(s)$ according to (23) so that DOB has perfect internal model compensation structure or equivalently, (18) is satisfied. Then, the optimal Q-filter is obtained by (15) and (10).

If $W_C(s)$ has at least one integrating factor, the designed DOB can perfectly suppress constant disturbance. Tuning free parameter in solving procedure of standard H_∞-control problem provides the possibility to attain controller with integral factors. But it is more desirable to reflect exactly integrator order specifications to be attained in $W_C(s)$ rather than tuning free parameter.

4.4. *Design of DOB guarantying robust stability of closed-loop system*

Q-filter design problem (7) and its optimization algorithm in Sections 2 and 3 provide comprehensive analysis and systematic design strategy for designing DOB. It only satisfies the sufficient robust stability condition of DOB inner loop but not that of closed-loop system, because it uses the complementary sensitivity function T_{DOB} and the robust stability sufficient condition (4) for DOB inner loop. In order to consider the disturbance suppression performance and robust stability of the whole closed-loop system, its sensitivity and complementary sensitivity functions (6) must be used in the cost function

$$\max \gamma, \min_{\substack{Q(s)\in\Omega_k \\ Q(s)\in RH_\infty}} \left\|\begin{bmatrix}\gamma W_S(s)S_{clo}(s)\\ W_T(s)T_{clo}(s)\end{bmatrix}\right\|_\infty < 1. \tag{29}$$

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Note that if the feedback controller $C(s)$ can stabilize the closed-loop system without DOB, then $Q(s)$ satisfies that $Q(s) \in RH_\infty$ can stabilize the closed-loop system with DOB. This can be easily verified from the fact that the stability of closed-loop system with DOB can be translated into the stability of $T_0 = P_n C/(1 + P_n C)$ (complementary sensitivity function without DOB) and that of $Q(s)$ itself from (6). Design problem (29) cannot use the solving algorithm developed in Section 3 because of the difference in structure between $T_{DOB}(s)$ and $T_{clo}(s)$ as well as that of $S_{DOB}(s)$ and $S_{clo}(s)$. The method for transforming this problem into the form of (7) is developed in this subsection. The robust stability condition reflected in (29) is

$$\|W_T(s)T_{clo}(s)\|_\infty < 1, \tag{30}$$

which could be rewritten as

$$|L(j\omega) + Q(j\omega)| < |W_T^{-1}(j\omega)(1 + L(j\omega))|, \forall\omega, \tag{31}$$

where $L(s) := P_n(s)C(s)$ is the open-loop transfer function of feedback loop. Because

$$|L(j\omega) + Q(j\omega)| < |L(j\omega)| + |Q(j\omega)|, \forall\omega, \tag{32}$$

Equation (31) always holds if

$$|Q(j\omega)| < |W_T^{-1}(j\omega)(1 + L(j\omega))| - |L(j\omega)|, \forall\omega.$$

Select a stable weighting function $W_{TD}(s)$ such that

$$E(\omega) := |W_T^{-1}(j\omega)(1 + L(j\omega))| - |L(j\omega)|, |W_{TD}^{-1}(j\omega)| < E(\omega), \forall\omega, \tag{33}$$

then the robust stability condition can be expressed as

$$|Q(j\omega)| < |W_{TD}^{-1}(j\omega)|, \forall\omega. \tag{34}$$

It could be easily be proved that frequency function $E(\omega)$ satisfies $E(\omega) > 0, \forall\omega$, if the system satisfies general robust stability condition $\|W_U(s)T_0(s)\|_\infty < 1$, where $T_0(s) = L(s)(1+L(s))^{-1}$ is complementary sensitivity function of the system without DOB. In general, robust stability defined by partitioning of absolute value of complex number summation such as (32) may be somehow conservative. We can rewrite the robust stability and sensor noise rejection condition as

$$\|W_{TD}(s)T_{DOB}(s)\|_\infty = \|W_{TD}(s)Q(s)\|_\infty < 1. \tag{35}$$

On the other hand, according to disturbance rejection performance of (29), we obtain

$$\left|\gamma \frac{W_S(j\omega)}{1 + L(j\omega)}(1 - Q(j\omega))\right| < 1, \forall\omega, \tag{36}$$

which can be regarded as a constraint on $S_{DOB}(s) = 1 - Q(s)$. If we select a stable weighting function $W_{SD}(s)$ such that

$$\left|\frac{W_S(j\omega)}{1 + L(j\omega)}\right| < |W_{SD}(j\omega)|, \forall\omega, \tag{37}$$

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the disturbance suppression optimization problem can be rewritten as

$$\max \gamma, \min_{Q(s)} \|\gamma W_{SD}(s)(1 - Q(s))\|_\infty < 1. \tag{38}$$

Finally, by (35) and (38), we replace the design problem (29) for the closed-loop system by a Q-filter design problem

$$\max \gamma, \min_{\substack{Q(s)\in\Omega_{n,k,q} \\ Q(s)\in RH_\infty}} \left\| \begin{bmatrix} \gamma W_{SD}(s)(1 - Q(s)) \\ W_{TD}(s)Q(s) \end{bmatrix} \right\|_\infty < 1, \tag{39}$$

where $\Omega_{n,k,q}$ is the set of rational functions satisfying relative order condition (8) and internal model order condition (18), that is,

$$\Omega_{n,k,q} = \left\{ G(s)|G(s) = \frac{M(s)}{N(s)}, N(s) = \sum\nolimits_{i=0}^{n} a_i s^i, M(s) = \sum\nolimits_{j=0}^{m} b_j s^j, \right.$$
$$\left. a_n \neq 0, b_m \neq 0, k = n - m, a_l = b_l, l = 0, 1, \cdots, q \right\}.$$

Notice that (39) is in the form of problem (7). It means that original design problem (29) with closed-loop system can be transformed into design problem (7) by suitable selection of the weighting functions. Thus, we can obtain the optimal Q-filter by systematic and straightforward algorithm of standard H_∞ control framework shown in Section 3. Especially, we can easily realize additional desired order structure of Q-filter, such as the whole order, relative order as well as internal model order.

5. DESIGN EXAMPLES

Consider a Q-filter of order 3 and relative order 2, the form of which is most frequently employed in servo control systems. The order specification of the filter design is whole order (denominator order) $n = 3$, numerator order $m = 1$, and relative order $k = n - m = 2$. Let internal model order specification be $q = m = 1$, which is the highest possible order. Therefore, we can simply choose the weighting functions as

$$\bar{W}_C(s) = \gamma W_C(s) = \gamma/(s + \lambda)^2, W_Q(s) = s^2/\alpha, \tag{40}$$

where $X = 0.0001$ is the number mentioned in (21) and $\alpha = 667$ is used to specify the cutoff frequency of Q-filter. Select a virtual plant satisfying Assumption 1 for given relative order as

$$\tilde{P}(s) = 1/(s + \delta)^2, \tag{41}$$

with $\delta = 3$ arbitrarily defined.

We can use MATLAB Control System Toolbox and Robust Control Toolbox to solve this problem. Especially, we use *hinfopt* function of Robust Control Toolbox to realize optimization through γ-iteration. The resultant optimal solution $\tilde{K}^*_{opt}(s)$ at $\gamma_{\max} \approx 150.203$, corresponding integrating controller $\tilde{K}^*_{opt}(s)$ in (23). And finally, the obtained optimal Q-filter is expressed as

$$\tilde{K}^*_{opt}(s) = \frac{666.7(s + 8.444)(s + 3)^2}{(s + 37.48)(s + 0.001)^2} = \frac{1}{(s + 0.001)^2 \tilde{K}^*_{\lambda,0}(s)},$$

$$\tilde{K}^*_{\lambda}(s) = \frac{1}{s^2}\tilde{K}^*_{\lambda,0}(s) = \frac{666.7(s + 8.444)(s + 3)^2}{(s + 37.48)s^2},$$

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$$Q^*(s) = \frac{\tilde{K}^*_\lambda(s)\tilde{P}(s)}{1+\tilde{K}^*_\lambda(s)\tilde{P}(s)} = \frac{666.7s+5632}{s^3+37.48s^2+666.7s+5632}.$$

The result satisfies the given demands of relative order and internal model order. The frequency magnitude responses of $Q^*(s)$, $W_Q^{-1}(s)$, $Q_C^*(s)$, and $W_C^{-1}(s)$ are shown in Figures 3 and 4. It is clear that all norm restriction demands in (7) are satisfied. In order to compare with previous traditional Q-filter, consider a binomial coefficient model filter as follows:

$$Q_b(s) = \frac{3(\sigma s)+1}{(\sigma s)^3+3(\sigma s)^2+3(\sigma s)+1}. \tag{42}$$

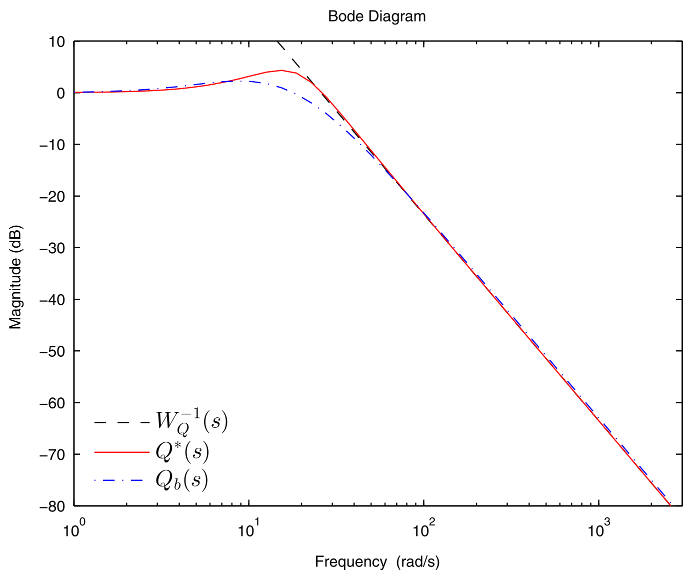

Figure 3. Frequency magnitude responses of $W_Q^{-1}(s)$, $Q^*(s)$ and $Q_b(s)$.

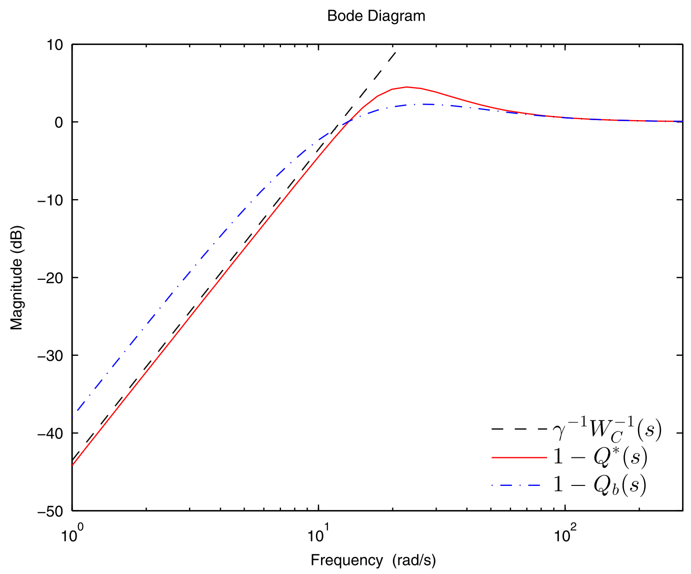

Figure 4. Frequency magnitude responses of $\gamma^{-1}W_C^{-1}(s)$, $1-Q^*(s)$ and $1-Q_b(s)$.

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Select the time constant as $\sigma = 0.065$ so that its roll-off frequency response coincides with $Q^*(s)$, meaning that two noise rejection performances are equal. The high frequency responses of $Q^*(s)$ and $Q_b(s)$ are shown in Figure 3. Through simple computations, we can know $|W_Q^{-1}(j\omega)| < |Q_b(j\omega)|$ for $\omega \geqslant \omega_H = 70$ rad/s. This shows that noise rejection performance of $Q^*(s)$ is better than $Q_b(s)$ at this frequency band. Under this condition, we can verify $|1 - Q^*(j\omega)| < |1 - Q_b(j\omega)|$ at $\omega < \omega_L = 10$ rad/s in Figure 4 and the magnitude difference between them at sufficiently low frequency is approximately 6 dB, showing that $Q^*(s)$ has the low frequency disturbances attenuation performance 2.2 times stronger than $Q_b(s)$. If we focus on noises at $\omega \geqslant \omega_H$ and disturbances at $\omega \leqslant \omega_L$ in design of Q-filter, $Q^*(s)$ is better than $Q_b(s)$. These differences in disturbance suppression performance get lager when the order of the observer gets higher. For example, in the case of order 5 and relative order 1 (i.e., $n = 5, m = 4$), the magnitude difference between $1 - Q^*(s)$ and $1 - Q_b(s)$ reaches 22.3 dB, which means that the former disturbance attenuation performance at low frequencies is 13 times stronger than the latter one.

6. APPLICATION TO CONTROL OF HARD DISK DRIVE

Positioning control of hard disk drive is considered in this section. The reference input is a step signal of 50 μm. The objective is to maintain the head at the reference point of disk in the presence of external disturbances and system uncertainties. The periodic disturbance with frequency of 43 Hz is used to show the comparison of disturbance suppression performance with the proposed method. The measurement noise is assumed as Gaussian distributed sensor noise. The LMI-based H_∞ robust DOB design method is employed for the aforementioned controlled plant [16]. In order to validate the effectiveness of the proposed method, we use the same conditions of model parameters, weighting functions, and controller as

$$\begin{cases} P(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2} \cdot G_i(s), \\ G_i(s) = \dfrac{(2.8 \cdot 10^4)^2}{s^2 + 6720s + (2.8 \cdot 10^4)^2}, \\ W_S(s) = \dfrac{0.5(s + 2000)}{s + 0.1}, \\ W_T(s) = \dfrac{1647.41s(s + 1.3 \cdot 10^4)^2}{(s + 10^6)(s + 5 \cdot 10^4)^2}, \\ C(s) = \dfrac{5850(0.00084s + 1)}{0.00011s + 1}, \end{cases} \tag{43}$$

where the first factor of $P(s)$ with $K = 0.002$, $\zeta_n = 0.25$, and $\omega_n = 600$ rad/s is regarded as nominal plant $P_n(s)$, and oscillation mode $G_i(s)$ has the frequency and damping factor with 28 000 rad/s and 0.12.

Then, from (33) and (37), we obtain the weighting functions in (39) as

$$\begin{cases} W_{SD} = \dfrac{0.5(s + 2000)}{s + 0.001}, \\ W_{TD} = \dfrac{(s + 100)(s^2 + 9 \cdot 10^8)}{10^{12}}. \end{cases} \tag{44}$$

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Hence, the designed optimal Q-filter is

$$Q^*(s) = \frac{9.999 \cdot 10^{11}}{s^3 + 6.35 \cdot 10^4 s^2 + 4.4 \cdot 10^8 s + 9.999 \cdot 10^{11}}. \tag{45}$$

From Figure 5, it is verified that robust stability condition (30) is satisfied. Figure 6 shows the comparison of disturbance suppression performance of $1 - Q^*(s)$ with the designed Q-filter $Q'(s)$ in the proposed method [16]. It is clear that the magnitude difference at low frequency is about 5 dB. Figure 7 shows tracking errors in the presence of periodic disturbance due to disk rotation. In the case of employing $Q'(s)$, maximum tracking error is 29 nm, whereas it is 17 nm when employing $Q^*(s)$. The tracking error in Figure 7 shows that the tracking control system with proposed DOB reduces the tracking error effectively.

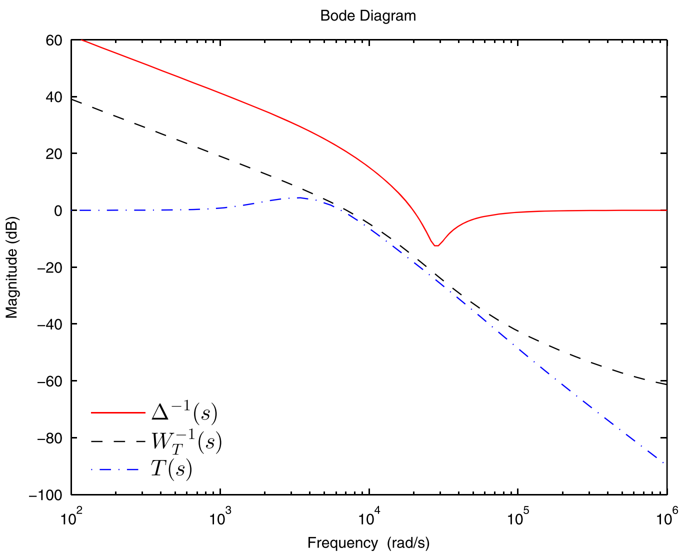

Figure 5. Verification of robust stability condition.

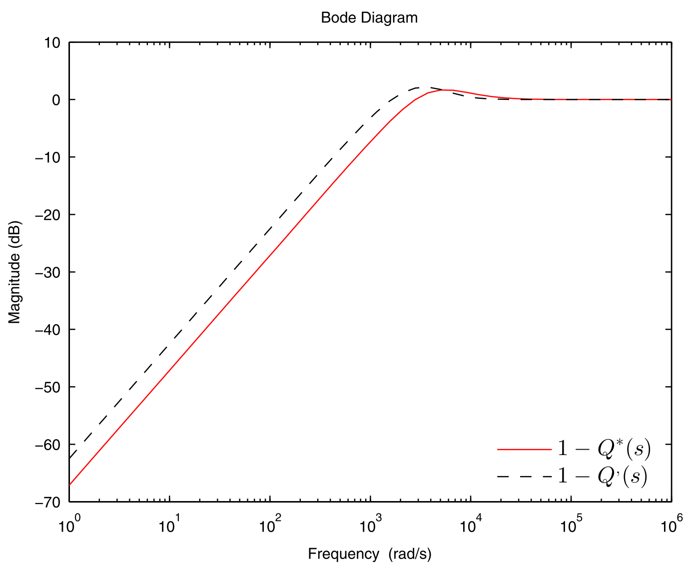

Figure 6. Comparison of disturbance suppression performance.

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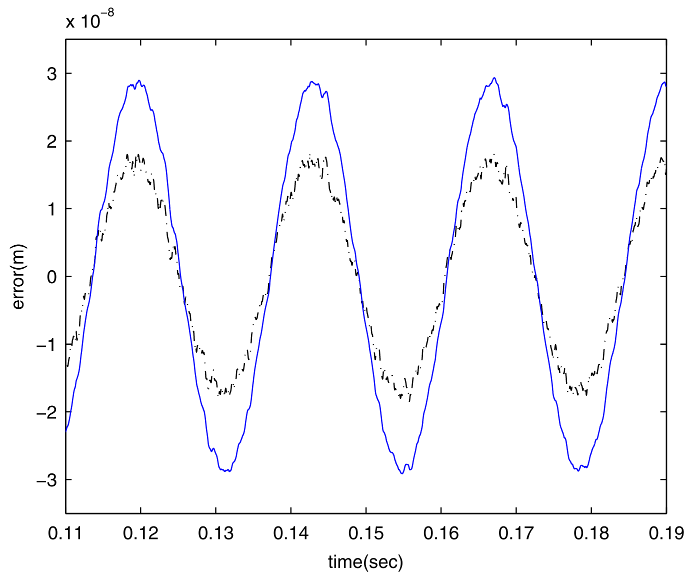

Figure 7. Time responses with periodic disturbance. Dashed line: proposed DOB in this paper; solid line: DOB in (Wang 2004).

7. CONCLUSIONS

In this paper, a design method of DOB based on H_∞-norm optimization has been presented, in which the standard H_∞ control framework is employed for solving optimization problem of DOB. The virtual factorization of loop transfer function of DOB system makes it possible for the Q-filter design procedure to be transformed to standard H_∞ control problem. Then, design specifications of Q-filter such as the order, relative order as well as internal model order are implemented by order and structure selection of virtual plant and weighting functions. Because the method presented in this paper provides a systematic and optimal design procedure, it can be regarded as generalized DOB design method. The numerical examples show the effectiveness and validity of the proposed method, and also, various design specifications can be implemented by different suitable selections of weighting functions under the condition of keeping the order structure.

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