Notes

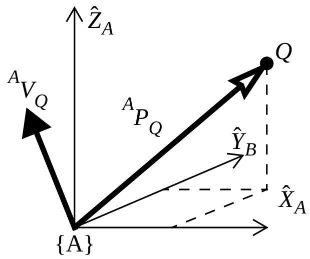
Table of Contents

Notes

Some Definitions

Linear Velocity:

Definition: The instantaneous rate-of-change in linear position of a point relative to some frame.



The position of a point Q in {A} is represented by the linear position vector, ${}^AP_Q$:

$$ {}^AP_Q = {}^A\begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^T \tag{1-1}$$

The velocity of a point Q relative to {A} is represented by the linear velocity vector, ${}^AV_Q$:

$$ {}^AV_Q \triangleq \frac{d}{dt}\left[{}^AP_Q\right] = {}^A\begin{bmatrix} \dot{q}_x & \dot{q}_y & \dot{q}_z \end{bmatrix}^T \tag{1-2}$$

LINEAR VELOCITY VECTOR

Angular Velocity:

Definition: The instantaneous rate-of-change in the orientation of one frame relative to another.

Just as there are many of ways to represent orientation (Euler Angles, Fixed Angles, Rotation Matrix, etc.), there are also many ways to represent the rate-of-change in orientation. We will focus on two: the Angular Velocity Vector and the Angular Velocity Matrix.

Angular Velocity Vector:

Definition: A vector whose direction is the instantaneous axis of rotation of one frame relative to another and whose magnitude is the rate of rotation about that axis.

The angular velocity vector,

$$ {}^A\Omega_B = \begin{bmatrix} \Omega_x & \Omega_y & \Omega_z \end{bmatrix}^T \tag{1-3}$$

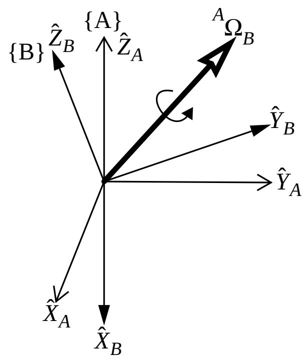
ANGULAR VELOCITY VECTOR

represents the instantaneous rate of rotation of {B} relative to {A}.

Angular Velocity Matrix:

The rotation matrix, ${}^A_BR$ defines the orientation of {B} relative to {A}. Specifically, the columns of ${}^A_BR$ are the unit vectors of {B} represented

Notes



in {A}:

$$ {}^A_BR = {}^A\left[\left[{}^BP_{\hat{X}}\right]\ \left[{}^BP_{\hat{Y}}\right]\ \left[{}^BP_{\hat{Z}}\right]\right] \tag{1-4}$$

If we look at the derivative of this rotation matrix, ${}^A_B\dot{R} = \frac{d}{dt}\left[{}^A_BR\right]$, the columns will be velocity of each unit vector of {B} with respect to {A}:

$$ {}^A_B\dot{R} = {}^A\left[\left[{}^BV_{\hat{X}}\right]\ \left[{}^BV_{\hat{Y}}\right]\ \left[{}^BV_{\hat{Z}}\right]\right] \tag{1-5}$$

Like the rotation matrix, this matrix has a number of "interesting" properties: Each column vector is perpendicular to the corresponding column vector in the rotation matrix and all three columns vectors are each perpendicular to a single vector which represents the instantaneous axis of rotation. This latter property implies some sort of relationship with the angular velocity vector since the instantaneous axis of rotation is a normalized angular velocity vector.

The relationship between ${}^A_B\dot{R}$ and ${}^A\Omega_B$ can be seen if we factor ${}^A_B\dot{R}$ into:

$$ {}^A_B\dot{R} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} {}^A_BR \tag{1-6}$$

The skew-symmetric matrix,

ANGULAR VELOCITY MATRIX

$$ {}^A_B\dot{R}_\Omega \triangleq \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \tag{1-7}$$

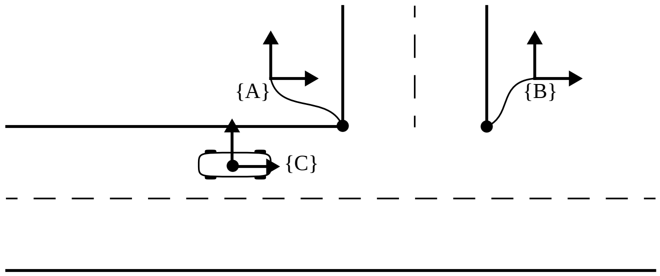
is called the angular velocity matrix and the elements of this matrix are the same as the elements of ${}^A\Omega_B = \left[\Omega_x\ \Omega_y\ \Omega_z\right]^T$.

Free Vectors:

Linear velocity vectors are insensitive to shifts in origin. In the example below, the velocity of the car frame {C} relative to both {A} and{B} is

Notes

the same. That is, ${}^{A}V_{C} = {}^{B}V_{C}$.



As long as {A} and {B} are fixed relative to each other and have the same orientation, the velocity is unchanged. For this reason, linear velocity vectors are called free vectors.

FREE VECTORS

Velocity Frames

When talking about the velocity (linear or angular) of an object, there are really two important frames that are being used:

1. The *frame of reference*: this is the frame used to measure the object's velocity
2. The *frame of representation*.: this is the frame in which the velocity is expressed.

VELOCITY FRAME OF REFERENCE

VELOCITY FRAME OF REPRESENTATION

For example, the linear velocity vector, ${}^{B}V_{Q}$ is the linear velocity of some point Q when using {B} as the frame of reference. If we transform this vector (e.g., ${}^{A}_{B}R\,{}^{B}V_{Q}$ then the resulting vector is *still* the velocity of Q relative to {B}, but *represented* in a different frame {A}.

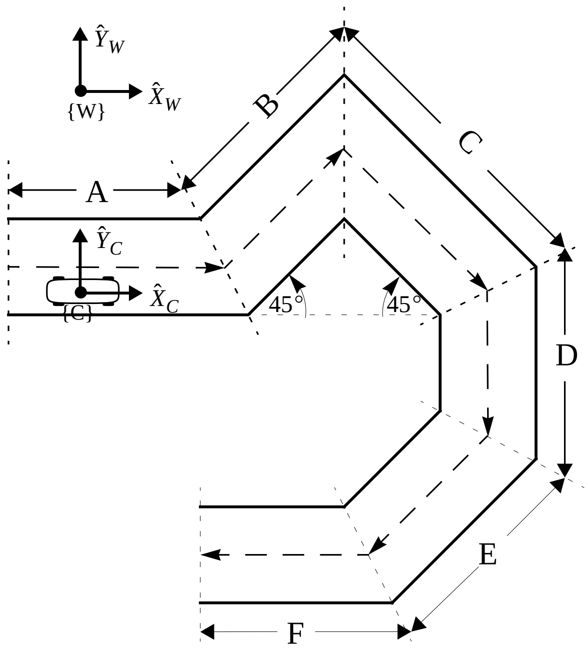
Our standard frame notation is augmented as follows to support this important distinction:

$$
\begin{aligned}
{}^{A}\left({}^{B}V_{Q}\right) &\triangleq {}^{A}_{B}R\,{}^{B}V_{Q} \\
{}^{A}\left({}^{B}\Omega_{C}\right) &\triangleq {}^{A}_{B}R\,{}^{B}\Omega_{C}
\end{aligned}
\qquad (1\text{-}8)
$$

Note that in the general case, ${}^{A}\left({}^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q} \neq {}^{A}V_{Q}$ because ${}^{A}_{B}R$ may be time-varying.

Notes

Using the figure below, and given that the car's (accurate) speedometer reads 100kph, fill in the elements of the table:



Road Section	Velocity			
	${}^C({}^C V_C)$	${}^W({}^W V_C)$	${}^W({}^C V_C)$	${}^C({}^W V_C)$
A	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$
B	$[0\ 0\ 0]^T$	$[71\ 71\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$
C	$[0\ 0\ 0]^T$	$[71\ -71\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$
D	$[0\ 0\ 0]^T$	$[0\ -100\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$
E	$[0\ 0\ 0]^T$	$[-71\ -71\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$
F	$[0\ 0\ 0]^T$	$[-100\ 0\ 0]^T$	$[0\ 0\ 0]^T$	$[100\ 0\ 0]^T$

Notes

Mathematics Toolbox

Angular Velocity: Vector vs. Matrix

The angular velocity vector and angular velocity matrix allow us to specify angular velocity in different ways. The vector form is convenient because it has an easy-to-grasp physical meaning; however, the matrix form is often more convent when doing algebraic manipulations. So we need to be able to shift easily from one form to the other as needed.

The following table shows a helpful list of corresponding matrix and vector forms:

Table 1: Matrix and Vector forms of angular velocity

${}^A_B\dot{R}_\Omega = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	$\longleftrightarrow$	${}^A\Omega_B = \begin{bmatrix} \Omega_x & \Omega_y & \Omega_z \end{bmatrix}^T$
$k\left({}^A_B\dot{R}_\Omega\right)$	$\longleftrightarrow$	$k\left({}^A\Omega_B\right)$
${}^A_B\dot{R}_\Omega \cdot \begin{bmatrix} x & y & z \end{bmatrix}^T$	$\longleftrightarrow$	${}^A\Omega_B \times \begin{bmatrix} x & y & z \end{bmatrix}^T$
${}^s_tR \cdot {}^A_B\dot{R}_\Omega \cdot {}^s_tR^T$	$\longleftrightarrow$	${}^s_tR \cdot {}^A\Omega_B$

Changing the Frame of Representation: Linear Velocity:

We have already used the homogeneous transform matrix, ${}^A_BT$, to compute the location of position vectors in other frames:

$$ {}^AP_Q = {}^A_BT\,{}^BP_Q \tag{1-9} $$

We use the derivative of this relation to study the relationship of velocity from one frame to another:

$$ \frac{d}{dt}\left[{}^AP_Q\right] = \frac{d}{dt}\left[{}^A_BT\,{}^BP_Q\right] \tag{1-10} $$

or

$$ {}^A\dot{P}_Q = {}^A_B\dot{T}\,{}^BP_Q + {}^A_BT\,{}^B\dot{P}_Q \tag{5-11} $$

where

$$ {}^A_B\dot{T} = \frac{d}{dt}\begin{bmatrix} \left[{}^A_BR\right] & \left[{}^AP_{Borg}\right] \\ 0\;0\;0 & 1 \end{bmatrix} = \begin{bmatrix} \left[{}^A_B\dot{R}_\Omega \cdot {}^A_BR\right] & \left[{}^AV_{Borg}\right] \\ 0\;0\;0 & 0 \end{bmatrix} \tag{5-12} $$

Notes

and $${}^{B}\dot{P}_Q = \frac{d}{dt}\begin{bmatrix} \begin{bmatrix} {}^{A}P_{Borg} \end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} {}^{A}V_{Borg} \end{bmatrix} \\ 0 \end{bmatrix}. \qquad (5\text{-}13)$$

Using (5-12) and (5-13), (5-11) can be expanded to

$$\begin{bmatrix} \begin{bmatrix} {}^{A}V_Q \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} {}^{A}_{B}\dot{R}_{\Omega} \cdot {}^{A}_{B}R \end{bmatrix} & \begin{bmatrix} {}^{A}V_{Borg} \end{bmatrix} \\ 0\ 0\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} {}^{B}P_Q \end{bmatrix} \\ 1 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} {}^{A}_{B}R \end{bmatrix} & \begin{bmatrix} {}^{A}P_{Borg} \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} {}^{B}V_Q \end{bmatrix} \\ 0 \end{bmatrix}. \qquad (1\text{-}14)$$

This expression can be broken down into the sum of three components:

$${}^{A}V_Q = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) + {}^{A}V_{Borg} + {}^{A}_{B}R\,{}^{B}V_Q \qquad (1\text{-}15)$$

We can convert the angular velocities in this expression to vector form using Table 1 on page 5 so that

$${}^{A}V_Q = {}^{A}\Omega_B \times \left({}^{A}_{B}R\,{}^{B}P_Q\right) + {}^{A}V_{Borg} + {}^{A}_{B}R\ {}^{B}V_Q. \qquad (1\text{-}16)$$

Note that in the special case where ${}^{A}_{B}\dot{R} = 0$ (i.e., ${}^{A}\Omega_B = 0$ and ${}^{A}V_{Borg} = 0$) then (1-16) simplifies to ${}^{A}V_Q = {}^{A}_{B}R\ {}^{B}V_Q = {}^{A}\left({}^{B}V_Q\right)$ which shows that ${}^{A}V_Q = {}^{A}\left({}^{B}V_Q\right)$ holds only in this special condition.

Changing the Frame of Reference: Angular Velocity

We use rotation matrices to represent angular position so that we can compute the angular position of {C} in {A} if we know the angular position of {C} in {B} and {B} in {A} by

$${}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R. \qquad (1\text{-}17)$$

To study how angular velocity propagates between frames, we will look at the derivative of (1-17):

$${}^{A}_{C}\dot{R} = {}^{A}_{B}\dot{R}\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R} \qquad (1\text{-}18)$$

Substituting in (1-6) yields

$${}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R. \qquad (1\text{-}19)$$

Post-multiplying by ${}^{A}_{C}R^{T}$ and simplifying yields

$${}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}. \qquad (1\text{-}20)$$

Using Table 1 on page 5, the angular velocities can also be expressed in vector form as:

$${}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R\,{}^{B}\Omega_C \qquad (1\text{-}21)$$

So the angular velocities of frames may be added as long as they are

Notes

expressed in the same frame.

Velocity propagation between robot links

The homogeneous transform matrix, ${}^{i-1}_{i}T$, provides a complete description of the linear and angular position relationship between adjacent robot links. These descriptions may be combined together to describe the position of a link relative to the robot base frame {0}:

$$ {}^{0}_{i}T = {}^{0}_{1}T\,{}^{1}_{2}T\ldots{}^{i-1}_{i}T \tag{1-22}$$

We would like to have a similar description of the linear and angular velocities between adjacent robot links and also between a given link and the robot base.

Frames and Notation:

In robotics, we are often interested in the velocity of a frame relative to the robot base. A special notation is used to express this:

$$ \begin{aligned} v_i &\triangleq {}^{0}\left({}^{0}V_i\right) \\ \omega_i &\triangleq {}^{0}\left({}^{0}\Omega_i\right) \end{aligned} \tag{1-23}$$

These velocities relative to the robot base are often expressed in other frames. The following notation is used for this:

$$ \begin{aligned} {}^{k}v_i &\triangleq {}^{k}\left({}^{0}V_i\right) = {}^{k}_{0}R \cdot v_i \\ {}^{k}\omega_i &\triangleq {}^{k}\left({}^{0}\Omega_i\right) = {}^{k}_{0}R \cdot \omega_i \end{aligned} . \tag{1-24}$$

Angular Velocity of adjacent links

From (1-21) we have the relation,

$$ {}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R\,{}^{B}\Omega_C \tag{1-25}$$

If we assign the link coordinate frames for adjacent links $\{i\}$ and $\{i+1\}$, with velocity computed relative to {0}, the robot base, we can make the following substitutions in (1-25):

$$ 0 \rightarrow A \qquad i \rightarrow B \qquad (i+1) \rightarrow C \tag{1-26}$$

so that (1-25) becomes

$$ {}^{0}\Omega_{i+1} = {}^{0}\Omega_i + \left({}^{0}_{i}R \cdot {}^{i}\Omega_{i+1}\right). \tag{1-27}$$

Here ${}^{i}\Omega_{i+1}$ is the relative rotation of the links due to joint rotation.

We can premultiply by ${}^{i+1}_{0}R$ to convert the frame-of-reference for the left side to $\{i+1\}$, yielding:

Notes

$$\left({}^{i+1}_{0}R \cdot {}^{0}\Omega_{i+1}\right) = \left({}^{i+1}_{0}R \cdot {}^{0}\Omega_{i}\right) + \left({}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}\Omega_{i+1}\right) \tag{1-28}$$

Substituting in the notation defined in (1-24) yields

$${}^{i+1}\omega_{i+1} = {}^{i+1}\omega_{i} + {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}\,. \tag{1-29}$$

The second term of this sum is the angular velocity of $\{i+1\}$ relative to $\{i\}$ and expressed in $\{i+1\}$, which is just $\begin{bmatrix}0 & 0 & \dot{\theta}_{i+1}\end{bmatrix}^{T}$. The first term can be rewritten in terms of ${}^{i}\omega_{i}$:

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \begin{bmatrix}0 & 0 & \dot{\theta}_{i+1}\end{bmatrix}^{T}}, \tag{1-30}$$

Equation (1-30) is recursive: it shows the angular velocity of one link in terms of the previous link as well as the relative motion of the two links. Since ${}^{i+1}\omega_{i+1}$ depends on all previous links through this recursion, angular velocity is said to "propagate" from the base to subsequent links.

Linear velocity of adjacent links

From (1-15), we have the relation,

$${}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{Borg} + {}^{A}_{B}R\,{}^{B}V_{Q}\,. \tag{1-31}$$

If we assign the link coordinate frames for adjacent links $\{i\}$ and $\{i+1\}$, with velocity computed relative to {0}, the robot base, we can make the following substitutions in (1-31):

$$0 \to A \qquad i \to B \qquad (i+1) \to Q \tag{1-32}$$

so that (1-31) becomes

$${}^{0}V_{(i+1)} = {}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{(i+1)}\right) + {}^{0}V_{i} + {}^{0}_{i}R\,{}^{i}V_{(i+1)} \tag{1-33}$$

Pre-multiplying by ${}^{i+1}_{0}R$ yields

$${}^{i+1}_{0}R\;{}^{0}V_{(i+1)} = {}^{i+1}_{0}R\;{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\;{}^{i}P_{i+1} + {}^{i+1}_{0}R\;{}^{0}V_{i} + {}^{i+1}_{0}R\;{}^{0}_{i}R\;{}^{i}V_{(i+1)} \tag{1-34}$$

which simplifies to

$${}^{i+1}_{0}R\;{}^{0}V_{(i+1)} = {}^{i+1}_{0}R\;{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\;{}^{i}P_{i+1} + {}^{i+1}_{0}R\;{}^{0}V_{i} + {}^{i+1}_{i}R\,{}^{i}V_{(i+1)}\,. \tag{1-35}$$

Factoring out ${}^{i+1}_{i}R$ from the left side of the first two terms and substituting $\begin{bmatrix}0 & 0 & \dot{d}_{i+1}\end{bmatrix}^{T}$ for the third term yields:

$${}^{i+1}_{0}R\;{}^{0}V_{(i+1)} = {}^{i+1}_{i}R\left\{\left({}^{i}_{0}R\;{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\right){}^{i}P_{(i+1)} + {}^{i}_{0}R\;{}^{0}V_{i}\right\} + \begin{bmatrix}0 & 0 & \dot{d}_{i+1}\end{bmatrix}^{T}. \tag{1-36}$$

Using Table 1 on page 5, we can write this in vector notation as

Notes

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 & 0 & \dot{d}_{i+1} \end{bmatrix}^T. \tag{1-37}$$

Like (1-30) for angular velocity, equation (1-37) is recursive: it shows the linear velocity of one link in terms of the linear and angular velocity of the previous link as well as the relative linear motion of the two links. Since ${}^{i+1}v_{i+1}$ depends on all previous links through this recursion, linear velocity is said to "propagate" from the base to subsequent links.

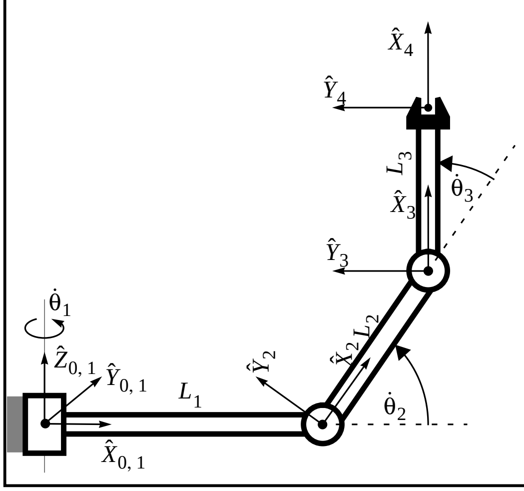
Summary and Example

The primary results of this section is the angular and linear velocity relationships between a given link and its previous link, as shown by (1-30) and (1-37) and repeated here:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 & 0 & \dot{\theta}_{i+1} \end{bmatrix}^T, \tag{1-38}$$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 & 0 & \dot{d}_{i+1} \end{bmatrix}^T. \tag{1-39}$$

We will use these relations to compute ${}^{4}\omega_4$ and ${}^{4}v_4$, the angular and linear velocity of the "tool" frame relative to {0} (the robot base) and expressed in {4} (the "tool" frame), for the following robot:



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$\pi/2$	L_1	0	θ_2
3	0	L_2	0	θ_3
4	0	L_3	0	0

Using techniques derived in earlier parts of this class, we can specify the homogeneous transform matrix ${}^{i}_{i+1}T$ for each adjacent link pair:

Notes

$$
{}^0_1T = \begin{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} \qquad
{}^1_2T = \begin{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix} & \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix}
$$

$$
{}^2_3T = \begin{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} \qquad
{}^3_4T = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{1-40}
$$

Using (1-38), we can first compute the per-link angular velocities:

$$
{}^1\omega_1 = {}^1_0R\,{}^0\omega_0 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \tag{1-41}
$$

$$
{}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} s_2\dot\theta_1 \\ c_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \tag{1-42}
$$

$$
{}^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_2\dot\theta_1 \\ c_2\dot\theta_1 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} s_{23}\dot\theta_1 \\ c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} \tag{1-43}
$$

$$
{}^4\omega_4 = {}^4_3\overset{I}{\cancel{R}}\,{}^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = {}^3\omega_3 = \begin{bmatrix} s_{23}\dot\theta_1 \\ c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} \tag{1-44}
$$

Now using (1-39), we can compute the per-link linear velocities (note that the $\begin{bmatrix} 0 & 0 & \dot d_{i+1} \end{bmatrix}^T$ term has been ignored since the robot in this example has no prismatic joints):

$$
{}^1v_1 = {}^1_0R\left({}^0\omega_0 \times {}^0P_1 + {}^0v_0\right) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1-45}
$$

$$
{}^2v_2 = {}^2_1R\left({}^1\omega_1 \times {}^1P_2 + {}^1v_1\right) = \begin{bmatrix} c_2 & 0 & s_2 \\ -s_2 & 0 & c_2 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot\theta_1 \end{bmatrix} \tag{1-46}
$$

Notes

$$
{}^{3}v_3 = {}^{3}_{2}R\left({}^{2}\omega_2 \times {}^{2}P_3 + {}^{2}v_2\right) = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s_2\dot{\theta}_1 \\ c_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}\right)
$$

$$
= \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ L_2\dot{\theta}_1 \\ -L_2c_2\dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} (L_2s_3)\,\dot{\theta}_2 \\ (L_2c_3)\,\dot{\theta}_2 \\ (-L_1 - L_2c_2)\,\dot{\theta}_1 \end{bmatrix} \qquad (1\text{-}47)
$$

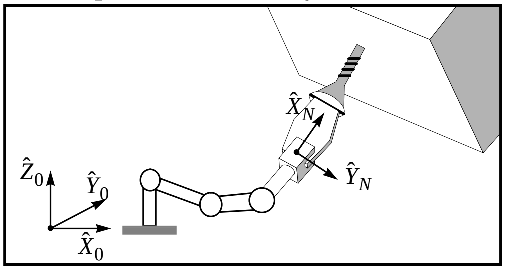
$$
{}^{4}v_4 = \overset{I}{\cancel{{}^{4}_{3}R}}\left({}^{3}\omega_3 \times {}^{3}P_4 + {}^{3}v_3\right) = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \times \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (L_2s_3)\,\dot{\theta}_2 \\ (L_2c_3)\,\dot{\theta}_2 \\ (-L_1 - L_2c_2)\,\dot{\theta}_1 \end{bmatrix}
$$

$$
= \begin{bmatrix} (L_2s_3)\,\dot{\theta}_2 \\ (L_2c_3 + L_3)\,\dot{\theta}_2 + L_3\dot{\theta}_3 \\ (-L_1 - L_2c_2 - L_3c_{23})\,\dot{\theta}_1 \end{bmatrix} \qquad (1\text{-}48)
$$

Notes

Robot Velocity

In robotics we are often called upon to manipulate objects with the tip of the robot which is represented here by frame {N}.



Kinematic Relations

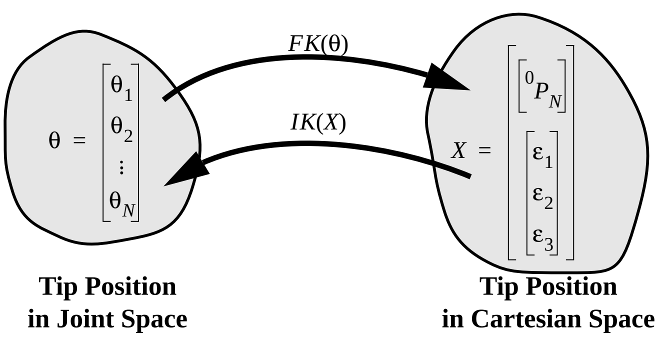
The location of the robot tip may be specified using either a joint space description,

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} \tag{1-49}$$

or using a cartesian space description,

$$X = \begin{bmatrix} \begin{bmatrix} {}^0P_N \end{bmatrix} \\ \begin{bmatrix} {}^0r_N \end{bmatrix} \end{bmatrix} \tag{1-50}$$

where ${}^0r_N$ is some 3-tuple description of the orientation of {N} relative to {0} using Euler Angles, Fixed Angles, or some other representation.



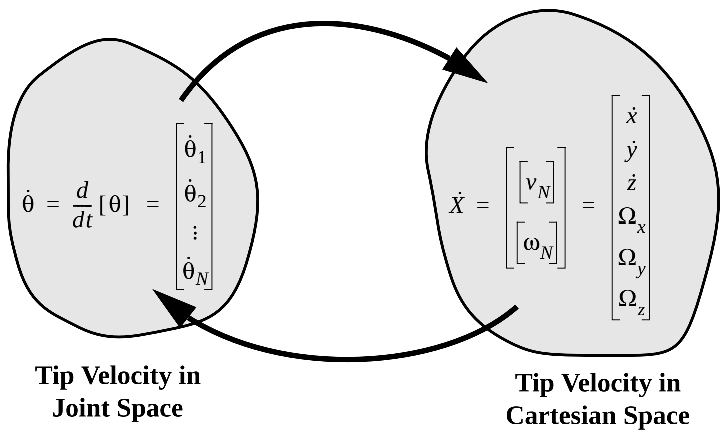
The robot kinematics equations relate these two descriptions of the robot tip position. Specifically, the forward kinematics, $X = FK(\theta)$, allows us to determine the cartesian position given the joint positions, while the inverse kinematics, $\theta = IK(X)$ allows us to determine the necessary joint positions to achieve a desired cartesian position.

Notes

Velocity Relations

Just as the cartesian position of the robot tip is interesting, so also is the cartesian velocity. Looking at the figure above, the necessary motions for inserting the screw can be described as a positive angular velocity about the x-axis of {N} while there is a small, positive linear velocity along the x-axis of {N} as the screw is driven in.

We would like to have mathematical expressions which do for joint velocity and cartesian velocities what forward and inverse kinematics do for joint and cartesian position:



The Robot Jacobian

We have already used the recursive expressions for adjacent joint linear/angular velocity

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 & 0 & \dot{\theta}_{i+1} \end{bmatrix}^T, \qquad (1\text{-}51)$$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 & 0 & \dot{d}_{i+1} \end{bmatrix}^T. \qquad (1\text{-}52)$$

to compute ${}^{N}\omega_N$ and ${}^{N}v_N$ for our $(N = 4)$ example robot, yielding:

$$ {}^{4}v_4 = \begin{bmatrix} (L_2 s_3)\,\dot{\theta}_2 \\ (L_2 c_3 + L_3)\,\dot{\theta}_2 + L_3\dot{\theta}_3 \\ (-L_1 - L_2 c_2 - L_3 c_{23})\,\dot{\theta}_1 \end{bmatrix} \text{ and } {}^{4}\omega_4 = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \qquad (1\text{-}53)$$

Looking at the figure above, the ${}^{4}\omega_N$ and ${}^{4}v_N$ values are exactly what we want to relate $\dot{X}$ to $\dot{\theta}$:

Notes

$$ {}^4\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \triangleq \begin{bmatrix} {}^4v_4 \\ {}^4\omega_4 \end{bmatrix} = \begin{bmatrix} (L_2 s_3)\,\dot{\theta}_2 \\ (L_2 c_3 + L_3)\,\dot{\theta}_2 + L_3\dot{\theta}_3 \\ (-L_1 - L_2 c_2 - L_3 c_{23})\,\dot{\theta}_1 \\ s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \tag{1-54} $$

If we factor out the vector $\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^T$ from this matrix, we get

$$ {}^4\dot{X} = {}^4\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + L_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}. \tag{1-55} $$

It turns out that v_N and ω_N will always be a linear combination of the joint velocity elements of $\dot{\theta}$. So the expression in (1-54) will always be factorable into the form in (1-55).

We call the matrix,

$$ {}^4J(\theta) = {}^4\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + L_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{1-56} $$

the robot's Jacobian matrix, expressed in frame {4}. We can use the recursive equations in (1-51) and(1-52) to find ${}^N v_N$ and ${}^N\omega_N$ and thereby find the $(6 \times N)$ ${}^N J(\theta)$ matrix for any robot.

Properties of the Jacobian

Frame of Representation

In the example above, we derived ${}^N J(\theta)$ for our example robot; this matrix gives the cartesian velocity of the robot tip relative to the robot base and represented in frame $\{N = 4\}$. We write this as

$$ {}^4\dot{X} = {}^4J(\theta)\dot{\theta} \tag{1-57} $$

Notes

where

$$ {}^4\dot{X} = {}^4\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} {}^4v_N \\ {}^4\omega_N \end{bmatrix} \text{ and } \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix} \quad (1\text{-}58) $$

It would be useful to know how to convert the Jacobian matrix so that the cartesian velocity, $\dot{X}$ is expressed in some other frame. We know that since

$$ {}^B\dot{X} \triangleq \begin{bmatrix} {}^Bv_N \\ {}^B\omega_N \end{bmatrix}, \quad (1\text{-}59) $$

then

$$ {}^A\dot{X} = \begin{bmatrix} {}^Av_N \\ {}^A\omega_N \end{bmatrix} = \begin{bmatrix} {}^A_BR \cdot {}^Bv_N \\ {}^A_BR \cdot {}^B\omega_N \end{bmatrix}. \quad (1\text{-}60) $$

In terms of the Jacobian relation, $\dot{X} = J(\theta)\dot{\theta}$, we can express this change in representation as

$$ {}^A\dot{X} = {}^AJ(\theta)\dot{\theta} = \begin{bmatrix} \begin{bmatrix} {}^A_BR \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} {}^A_BR \end{bmatrix} \end{bmatrix} {}^BJ(\theta)\dot{\theta} \quad (1\text{-}61) $$

so that

$$ {}^AJ(\theta) = \begin{bmatrix} \begin{bmatrix} {}^A_BR \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} {}^A_BR \end{bmatrix} \end{bmatrix} {}^BJ(\theta) \quad (1\text{-}62) $$

is the relation which to converts a Jacobian from one frame of representation to another.

Using the result in (1-62), we can compute ${}^0J(\theta)$ for our example robot from ${}^4J(\theta)$ given in (1-56) by

Notes

$$
{}^0J(\theta) = \begin{bmatrix} \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix} & \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix} \\ \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix} & \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix} \end{bmatrix} {}^4\begin{bmatrix} 0 & L_2s_3 & 0 \\ 0 & L_2c_3+L_3 & L_3 \\ -(L_1+L_2c_2+L_3c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{1-63}
$$

$$
= \begin{bmatrix} -L_3s_1c_{23}-L_2s_2c_2-L_1s_1 & -L_3c_1s_{23}-L_2c_1s_2 & -L_3c_1s_{23} \\ L_3c_1c_{23}-L_2c_{12}+L_1c_1 & -L_3s_1s_{23}-L_2s_{12} & -L_3s_1s_{23} \\ 0 & L_3c_{23}+L_2c_3 & L_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}
$$

The two Jacobians, ${}^4J(\theta)$ and ${}^0J(\theta)$, both give us the cartesian velocity of the robot tip relative to the robot base

$$
\begin{aligned} {}^4\dot{X} &= {}^4J(\theta)\dot{\theta} \\ {}^0\dot{X} &= {}^0J(\theta)\dot{\theta} \end{aligned} \tag{1-64}
$$

but ${}^4J(\theta)$ gives us this velocity represented in {4} while ${}^0J(\theta)$ gives us this velocity represented in frame {0}.

Alternative approach to finding the Jacobian

The ${}^0_NT$ matrix contains all the necessary information to describe the position and orientation of frame {N} relative to frame {0}. Therefore, the derivative of this matrix, ${}^0_N\dot{T}$ will contain all the necessary information to describe the rate-of-change in position and orientation (i.e., linear and angular velocity) of frame {N} relative to frame {0} and expressed in frame {0}. This is the exact information that we need to express ${}^0J(\theta)$.

The contents of ${}^0_N\dot{T}$ will have the following structure:

$$
{}^0_N\dot{T} = \frac{d}{dt}\left[{}^0_NT\right] = \begin{bmatrix} \left[{}^0_N\dot{R}\right] & \left[{}^0\dot{P}_N\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \left[{}^0_N\dot{R}_\Omega \cdot {}^0_NR\right] & \left[{}^0V_N\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} \tag{1-65}
$$

If we postmultiply ${}^0_N\dot{T}$ by a special matrix which "cancels" the ${}^0_NR$ term in the rotation submatrix, we get a new matrix,

$$
{}^0_N\tilde{T} = \begin{bmatrix} \left[{}^0_N\dot{R}_\Omega \cdot {}^0_NR\right] & \left[{}^0V_N\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^0_NR^T\right] & \begin{bmatrix}0\\0\\0\end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \left[{}^0_N\dot{R}_\Omega\right] & \left[{}^0V_N\right] \\ 0\ 0\ 0 & 0 \end{bmatrix} \tag{1-66}
$$

Notes

which, by definition, is structured with the following elements:

$$ {}^{0}_{N}\tilde{T} = \begin{bmatrix} \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} & \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \\ 0\ 0\ 0 & 0 \end{bmatrix} \tag{1-67} $$

So, to find ${}^{0}J(\theta)$ for any mechanism, we could do the following:

1. Compute the following matrix using ${}^{0}_{N}T$ (and ${}^{0}_{N}R$ from ${}^{0}_{N}T$):

$$ {}^{0}_{N}\tilde{T} = \begin{bmatrix} \begin{bmatrix} {}^{0}_{N}\dot{R}_{\Omega} \cdot {}^{0}_{N}R \end{bmatrix} & \begin{bmatrix} {}^{0}V_N \end{bmatrix} \\ 0\ 0\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} {}^{0}_{N}R^T \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} {}^{0}_{N}\dot{R}_{\Omega} \end{bmatrix} & \begin{bmatrix} {}^{0}V_N \end{bmatrix} \\ 0\ 0\ 0 & 0 \end{bmatrix} \tag{1-68} $$

2. Using the pattern shown in (1-67), extract the following elements from this ${}^{0}_{N}\tilde{T}$ matrix:

$$ {}^{0}\dot{X} = {}^{0}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} {}^{0}_{N}\tilde{T}\langle 1, 4\rangle \\ {}^{0}_{N}\tilde{T}\langle 2, 4\rangle \\ {}^{0}_{N}\tilde{T}\langle 3, 4\rangle \\ {}^{0}_{N}\tilde{T}\langle 3, 2\rangle \\ {}^{0}_{N}\tilde{T}\langle 1, 3\rangle \\ {}^{0}_{N}\tilde{T}\langle 2, 1\rangle \end{bmatrix} \tag{1-69} $$

3. Factor out the $\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dots & \dot{\theta}_N \end{bmatrix}^T$ from (1-69) to form the velocity relation

$$ \dot{X} = {}^{0}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = {}^{0}J(\theta) \cdot \dot{\theta} \tag{1-70} $$

The computation of ${}^{0}J(\theta)$ by this approach is often more difficult than the recursive approach shown earlier, but, as a second method to find the Jacobian, it can provide a some additional confidence that a given Jacobian derivation is correct if the two approaches produce identical results (after rotating either ${}^{0}J(\theta)$ or ${}^{4}J(\theta)$ to match the frame of representation of the other).

Notes

The Inverse Jacobian Relation

We now have a the necessary tools to derive the Jacobian matrix for any robot in any desired frame of representation and compute the cartesian velocity of the robot tip represented in that frame using

$$\dot{X} = J(\theta)\dot{\theta}\,. \tag{1-71}$$

Can we solve for $\dot{\theta}$ in terms of $\dot{X}$? This would be very useful because it would tell us the joint velocities necessary to achieve a desired cartesian velocity. We can compute this relation from (1-71) as

$$\dot{\theta} = J^{-1}(\theta)\dot{X} \tag{1-72}$$

However, is $J^{-1}(\theta)$ invertible? In general, the $(6 \times N)$ Jacobian matrix may be non-square in which case the inverse is undefined. A matrix is invertible only if it has a non-zero determinant (non-invertible matrix is called a *Singular* matrix). Looking at the Jacobian, ${}^4J(\theta)$ and augmenting the robot with three zero-valued joints to make a square Jacobian matrix,

$${}^4\dot{X} = {}^4\begin{bmatrix} 0 & L_2 s_3 & 0 & 0 & 0 & 0 \\ 0 & L_2 c_3 + s_3 & L_3 & 0 & 0 & 0 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 & 0 & 0 & 0 \\ s_{23} & 0 & 0 & 0 & 0 & 0 \\ c_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1-73}$$

we can see immediately that the matrix is *not* invertible because one or more rows or columns of zeros means that the determinant is zero.

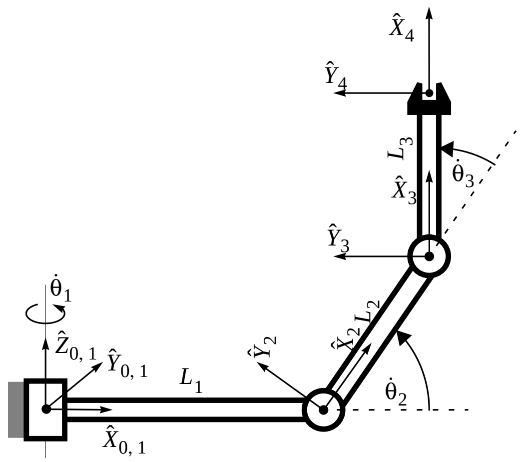
The Reduced Jacobian

We know that a $N < 6$ manipulator does not have the necessary degrees of freedom to achieve independent control of its all six cartesian velocity components, $\dot{X} = \begin{bmatrix}\dot{x} & \dot{y} & \dot{z} & \Omega_x & \Omega_y & \Omega_z\end{bmatrix}^T$. We can see this limitation in the Jacobian relation for our example robot:

$${}^4\dot{X} = {}^4\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = {}^4\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + s_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \tag{1-74}$$

we can see that the Jacobian provides a system of six equations for the six unknown elements of $\dot{X}$ which are all dependant on the $(N = 3)$

Notes



robot joint variables. This underdetermined system of equations must have interdependencies among the elements of $\dot{X}$.

In this particular case, the three shaded rows of ${}^4J(\theta)$ are linearly dependent since each can be expressed as a scaled version of another. This linear dependence implies that only one of the three shaded elements of ${}^4\dot{X}$, $(\dot{z}, \Omega_x, \Omega_y)$, can be independently specified. For the remaining three (unshaded) rows, any one row is linearly dependant on the other two. This means that only two of the three unshaded elements of ${}^4\dot{X}$, $(\dot{x}, \dot{y}, \Omega_z)$, can be independently specified.

In general, if $(N < 6)$, there will be only N independent rows in the Jacobian matrix no matter what frame of representation. Which N rows (and corresponding elements in $\dot{X}$) one chooses to call "independent" is a matter of choosing between a limited set of options.

In the case of the example robot, we could choose to consider a reduced cartesian vector,

$$ {}^4\dot{X}_r = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad (1\text{-}75) $$

as our independant axes of control in frame {4}.

Computing the Inverse Jacobian

If we reduce the number of rows in the original Jacobian to the rows corresponding to our choice of ${}^4\dot{x}_r$ in (1-75) above, then the resulting "reduced" Jacobian matrix will be square (there are exactly N independent rows in the Jacobian of an N-link manipulator).

For our example robot, the reduced Jacobian, ${}^4J_r(\theta)$ will be

Notes

$$ {}^{4}J_r(\theta) = {}^{4}\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + s_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \end{bmatrix} \tag{1-76} $$

and the cartesian velocity of the robot tip in our chosen reduced cartesian velocity vector will be

$$ {}^{4}\dot{X}_r \triangleq {}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + s_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} . \tag{1-77} $$

This reduced Jacobian matrix will always be square and therefore is at least *potentially* invertible so that we *may* be able to compute the robot joint velocity, $\dot{\theta}$ necessary to achieve some desired cartesian velocity, ${}^{4}\dot{X}_r$ using

$$ \dot{\theta} = {}^{4}J_r^{-1}(\theta) \cdot {}^{4}\dot{X}_r . \tag{1-78} $$

Robot Singular Configurations

If we want to use the relation in (1-78) to compute $\dot{\theta}$, we need to first find out at what points the inverse exists.

Since the matrix is only invertible when the it has a non-zero determinant, computing the determinant symbolically allows us to find the complete set of values of θ for which the Jacobian, ${}^{4}J(\theta)$ is singular. In the case of our example robot, we have

$$ {}^{4}J_r(\theta) = {}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2 s_3 & 0 \\ 0 & L_2 c_3 + s_3 & L_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) & 0 & 0 \end{bmatrix} \tag{1-79} $$

which has the determinant

$$ \left| {}^{4}J_r(\theta) \right| = -(L_1 + L_2 c_2 + L_3 c_{23})\ (L_2 s_3)\, L_3 \tag{1-80} $$

so that the matrix is singular (non-invertible) when

$$ -(L_1 + L_2 c_2 + L_3 c_{23})\ (L_2 s_3)\, L_3 = 0 . \tag{1-81} $$

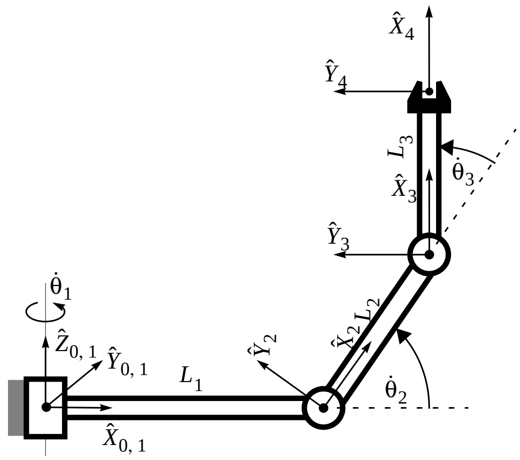
This singular condition occurs when either of the following are true:

$$ \begin{aligned} s_3 &= 0 \\ -L_1 - L_2 c_2 - L_3 c_{23} &= 0 \end{aligned} \tag{1-82} $$

If we examine the conditions in (1-82) along with the schematic of the

Notes

example robot below, we can see how each of these two types of "sin-



gular" positions may occur:

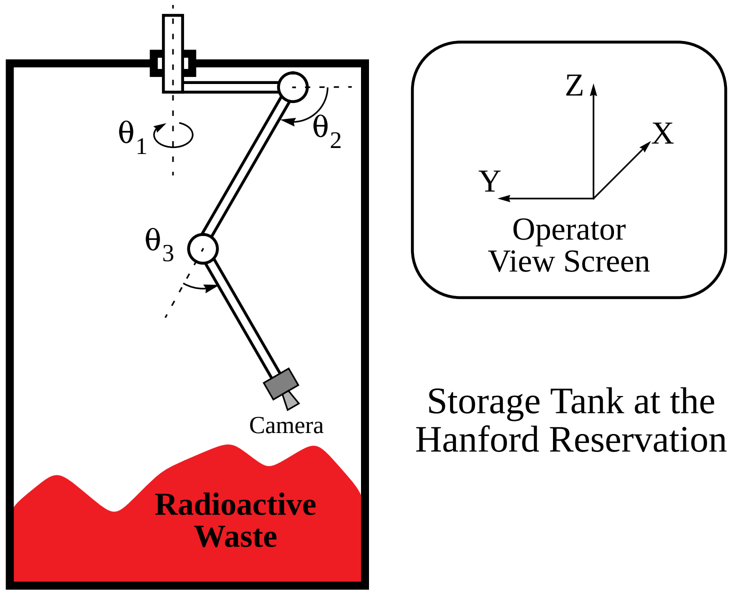
$(s_3 = 0) \rightarrow \begin{cases} \theta_3 = 0 \\ \theta_3 = 180° \end{cases}$	When $\theta_3 \in \{0°, 180°\}$, then the first row of ${}^4J_r(\theta)$ goes to zero and the robot cannot move along the X-axis of {4}
$-L_1 - L_2c_2 - L_3c_{23} = 0$	When the origin of {4} intersects the z-axis of {1}, the third row of ${}^4J_r(\theta)$ goes to zero and the robot cannot move along the Z-axis of {4}. This may only occur if $L_2 + L_3 \geq L_1$ is true.

Joint velocity near singular positions

Imagine that a robot with the same kinematic structure as our example robot is being used for a visual inspection task such as the one shown

Notes

below. The robot system is designed so that the operator can command

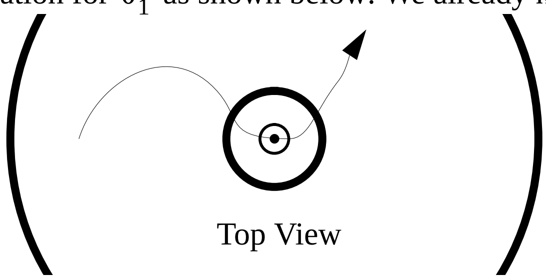


Storage Tank at the Hanford Reservation

the robot motions in frame {4} which is fixed relative to the camera view. A computer system interprets the operator's commands and computes the necessary joint velocities to achieve the commanded cartesian motion using the relation

$$\dot{\theta} = {}^4J_r^{-1}(\theta) \cdot \dot{X}_r \tag{1-83}$$

The operator, who has never heard of a Jacobian or a Singularity, inadvertently drives the robot tip slowly along a path which takes it the near the axis of rotation for θ_1 as shown below. We already know that the



Jacobian will be singular when the robot tip frame {4} intersects the axis of θ_1 and that the robot will be unable to move along the ${}^4\dot{z}$ axis since the third row of the Jacobian becomes all zeros. But what happens when the robot is *approaches* the singularity condition?

If we look at the third row of the Jacobian expressed in frame {4},

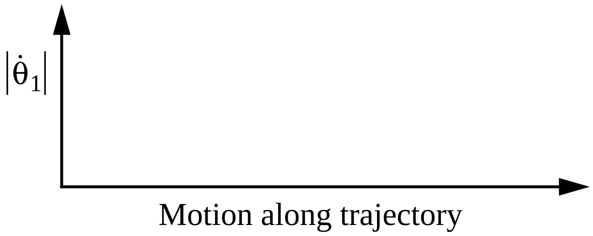
$${}^4\dot{z} = -(L_1 + L_2c_2 + L_3c_{23})\,\dot{\theta}_1 \tag{1-84}$$

and solve for $\dot{\theta}_1$ in terms of ${}^4\dot{z}$, we find that

$$\dot{\theta}_1 = \frac{-{}^4\dot{z}}{(L_1 + L_2c_2 + L_3c_{23})}\,. \tag{1-85}$$

Notes

So as we approach this singularity condition with *any* non-zero commanded $^{4}\dot{z}$ value, the value of $\dot{\theta}_1$ necessary to achieve this motion goes to infinity! The figure below show a profile of the joint velocity as the robot moves along this trajectory. This type of trajectory causes two



problems:

1. The robot is physically limited from unusually high joint velocities by motor power constraints, etc. So the robot will be unable to track this joint velocity trajectory exactly, resulting in some perturbation to the commanded cartesian velocity trajectory
2. The slopes of the joint velocity plot above represents the acceleration of this joint. The high accelerations that come from approaching too close to a singularity have caused the destruction of many robot gears and shafts over the years.