The Incremental Motion Model:

The Jacobian Matrix

In the forward kinematics model, we saw that it was possible to relate joint angles $\underline{\theta}$, to the configuration of the robot end effector ${}^{0}_{6}T$

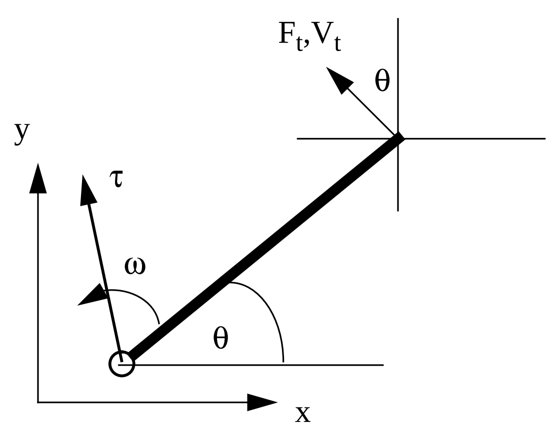
In this section, we will see that it is possible to model the relationship between the joint *rates*, $\dot{\theta}$, and the velocity of the end effector, $\dot{\underline{x}}$, with a matrix as follows:

$$\dot{\underline{x}} = J(\theta)\,\dot{\theta}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & & \\ & J(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dots \\ \dot{\theta}_N \end{bmatrix}$$

where ω_i is the *i*th component of angular velocity and $J(\theta)$ is a matrix of size 6 x N, and N is the number of joints on the robot. We will derive this matrix by calculating $\dot{\underline{x}}$ as a function of $\dot{\underline{\theta}}$ and factoring out $J(\theta)$. But first, as further illustration, let us consider a simple planar example.:



We can see from this example that we can resolve the velocity of the tip into x and y components as follows:

$$v_x = -r\omega\sin\theta$$

$$v_y = r\omega\cos\theta$$

which can be expressed in a trivial matrix equation as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \begin{bmatrix} \omega \end{bmatrix}$$

or

$$\underline{\dot{x}} = \mathrm{J}(\theta)\,\underline{\dot{\theta}}$$

Alternatively, we can look at the tip force, and the torque around the joint:

$$\tau = -rF_x\sin\theta + rF_y\cos\theta$$

which again gives a trivial matrix equation:

$$\tau = [-r\sin\theta, r\cos\theta] \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

or

$$\tau = \mathrm{J}^{\mathrm{T}}(\theta)\,\underline{F}$$

Jumping ahead from this simple example, we will see that the Jacobian Matrix can also be used to relate tip forces to joint torques.

Properties of Linear Transformations

Source: “Introduction to Matrices and Determinants,” F.Max Stein, 1967.

Facts:

1. A^{-1} exists if and only if

$$\det[A] \neq 0$$

and if $\det[A] = 0$, we say A is a *singular* matrix.

2. The *rank* of A is the size of the largest square sub-matrix, *S*, for which

$$\det[S] \neq 0$$

3. If two rows or columns of A are equal or related by a constant, then $\det[A] = 0$.

4. Any singular matrix has at least one eigenvalue equal to zero.

5. If A is non-singular, and λ is an eigenvalue of A corresponding to the eigenvector *x*, then

$$A^{-1}x = \lambda^{-1}x$$

6. If A is square, then A and A^T have the same eigenvalues: i.e.

$$Ax_i = \lambda_i x_i \qquad \text{and} \qquad A^T y_i = \lambda_i y_i$$

7. If the n x n matrix A is of full rank (i.e. $r = n$) then the only solution to $Ax = 0$ is the trivial one, $x = 0$.

8. If $r < n$, there are $n - r$ linearly independent (i.e. orthogonal) solutions,

$$x_j \qquad (1 \leq j \leq n - r)$$

for which $Ax_j = 0$ (see also 4.)

Properties of the Jacobian Matrix

We have seen

$$\dot{x} = J(\underline{\theta})\,\dot{\underline{\theta}} \qquad (1)$$

$$\text{and } \dot{\underline{\theta}} = J^{-1}(\underline{\theta})\,\underline{\dot{x}} \qquad (2).$$

Velocity Mapping

Consider the square 6x6 case. If $\mathrm{rank}(J(\underline{\theta})) < 6$, (2) has no solution (facts 1 & 2). Also, by fact 8, if the rank, r = 5, there is one non-trivial solution to

$$\dot{x} = J(\underline{\theta})\,\dot{\underline{\theta}} = 0$$

so there is a direction in joint velocity space for which joint motion produces no EE motion.

We call any joint configuration, $\underline{\theta} = Q$, for which

$$\mathrm{rank}(J(Q)) < 6$$

a **singular configuration**.

Furthermore, for certain directions of EE motion, $\dot{x}_i$, $1 \le i \le 6$,

$$\underline{\dot{x}}_i = J(\underline{\theta})\,\dot{\underline{\theta}} = \lambda_i(\underline{\theta})\,\underline{\omega}_i(\underline{\theta})$$

where

$\lambda_i(\underline{\theta})$ are eigenvalues of $J(\theta)$, and

$\underline{\omega}_i(\underline{\theta})$ are eigenvectors of $J(\underline{\theta})$

if $J(\underline{\theta})$ is full rank, we have:

$$\dot{\underline{\omega}}_i = J^{-1}(\underline{\theta})\,\underline{\dot{x}}_i = \lambda_i^{-1}\underline{\dot{x}}_i$$

(see fact 5). As we approach the singular configuration $\underline{\theta} = Q$, there is at least one eigenvalue (j) for which $\lambda_j \to 0$. Thus,

$$\dot{\underline{\omega}}_j \to \frac{\underline{\dot{x}}_j}{0} \to \infty$$

In other words, motion in the particular direction $\underline{\dot{x}}_j$ causes joint velocities to approach infinity.

Force Mapping

Consider $\tau = J^T(\underline{\theta})F$

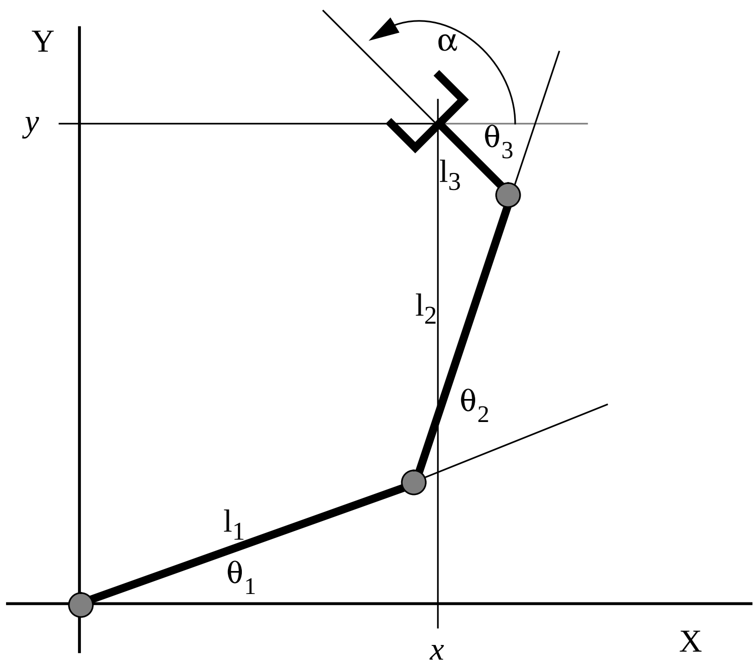
Clearly, $\text{rank}\left(J^T\right) = \text{rank}(J)$, since $\det[A] = \det\left[A^T\right]$. Thus, at $\underline{\theta} = Q$, there exists

$$F \neq 0 \quad \text{such that} \quad J^T(Q)F = 0$$

In other words, a finite force in a certain direction produces **no torque** at the joints (if $\text{rank}[J(Q)] = 5$, there is just one such force direction, see facts 4,6,7).

The Jacobian by Differentiation

In addition to the method of velocity propogation, the Jacobian can be obtained by differentiation of the forward kinematics equations.



Considering the above planar manipulator, define the configuration vector $\underset{\sim}{x} = [x, y, \alpha]^T$. The forward kinematic equations of this arm are:

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$\alpha = \theta_1 + \theta_2 + \theta_3$$

Differentiating the first of these expressions, gives

$$\dot{x} = -l_1 s_1 \dot{\theta}_1 - l_2 s_{12} (\dot{\theta}_1 + \dot{\theta}_2) - l_3 s_{123} (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

or

$$\dot{x} = -(l_1 s_1 + l_2 s_{12} + l_3 s_{123})\dot{\theta}_1 - (l_2 s_{12} + l_3 s_{123})\dot{\theta}_2 - (l_3 s_{123})\dot{\theta}_3$$

Similarly,

$$\dot{y} = (l_1 c_1 + l_2 c_{12} + l_3 c_{123})\,\dot{\theta}_1 + (l_2 c_{12} + l_3 c_{123})\,\dot{\theta}_2 + (l_3 c_{123})\,\dot{\theta}_{123}$$

and

$$\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

Thus,

$$^0\mathrm{J}(\underline{\theta}) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

Remember that we started with the expression for the EE configuration in frame 0 so the resulting Jacobian is still in frame 0. In contrast, the velocity propogation method leaves us with a Jacobian in frame N. Either result can be rotated later as necessary.

Example: Singular Configuration

To look at a singular configuration of this manipulator, take for example,

$$Q = \begin{bmatrix} \frac{\pi}{4} \\ 0 \\ \pi \end{bmatrix}$$

Note that for $\underline{\theta} = Q$,

$s_{12} = s_1$, $s_{123} = -s_1$, $c_{12} = c_1$, $c_{123} = -c_1$. This gives

$$^0\mathrm{J}(Q) = \begin{bmatrix} (-l_1 - l_2 + l_3)\,s_1 & (-l_2 + l_3)\,s_1 & l_3 s_1 \\ (l_1 + l_2 - l_3)\,c_1 & (l_2 - l_3)\,c_1 & -l_3 c_1 \\ 1 & 1 & 1 \end{bmatrix}$$

If we denote the rows of J by {r1,r2, r3}, and recall that for $\theta_1 = \frac{\pi}{4}$,

$s_1 = c_1 = 0.707$ then we note that

$$\mathrm{r1} = (-1)\,\mathrm{r2}$$

so that $\det[\mathrm{J}(Q)] = 0$ (fact 3).

This means that

$$Q = \begin{bmatrix} \frac{\pi}{4} \\ 0 \\ \pi \end{bmatrix}$$

is a singular configuration. Note that on closer examination, we see that $J(Q)$ is singular for **any** value of θ_1! (This should be obvious from the geometry). This is because, for any θ_1, we still have

$$r1 = r2(-\tan(\theta_1))$$

and $\tan(\theta_1)$ is a constant. Thus,

$$Q = \begin{bmatrix} \theta_1 \\ 0 \\ \pi \end{bmatrix}$$

is a singular configuration for any θ_1.

Force at a singularity

Since $\tau = J^T F$, at a singular point Q, we can expect non trivial (i.e. non zero) forces F_j such that

$$J^T(Q) F_j = 0$$

In words, there will be some force vector or vectors which can be applied to the tip which generate no torques in the joints. So, in a singular configuration, the mechanism can "lock up" with respect to tip forces or torques in certain directions. For example, suppose Q is defined as above, and

$$F_1 = \begin{bmatrix} -F\cos\theta_1 \\ -F\sin\theta_1 \\ 0 \end{bmatrix}$$

Note that this corresponds to a force applied to the tip in the direction opposite to the outstretched arm, and that no external torque is applied to the tip. Now, writing $J^T(Q)$ in a simplified form,

$$\underset{\sim}{\tau} = J^T(Q) F_1 = \begin{bmatrix} as_1 & -ac_1 & 1 \\ bs_1 & -bc_1 & 1 \\ cs_1 & -cc_1 & 1 \end{bmatrix} \begin{bmatrix} -Fc_1 \\ -Fs_1 \\ 0 \end{bmatrix}$$

where

$a = -l_1 - l_2 + l_3$

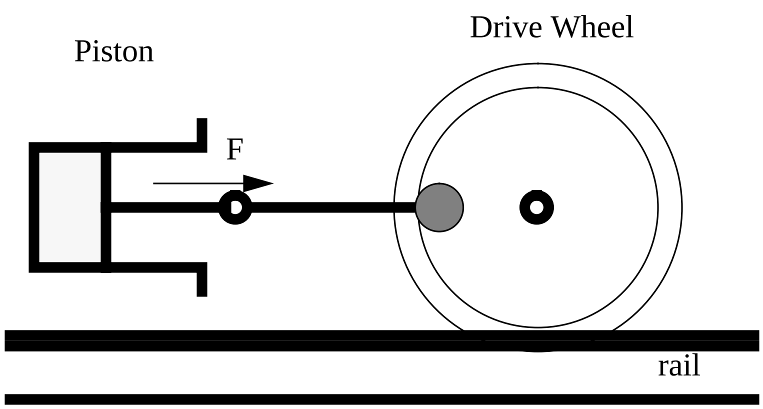
$b = -l_2 + l_3$

$c = l_3$

and

$$\underset{\sim}{\tau} = \begin{bmatrix} -aFs_1c_1 + aFs_1c_1 + 0 \\ -bFs_1c_1 + bFs_1c_1 + 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This situation is an old and famous one in mechanical engineering. For example, in the steam locomotive, "top dead center" refers to the following condition



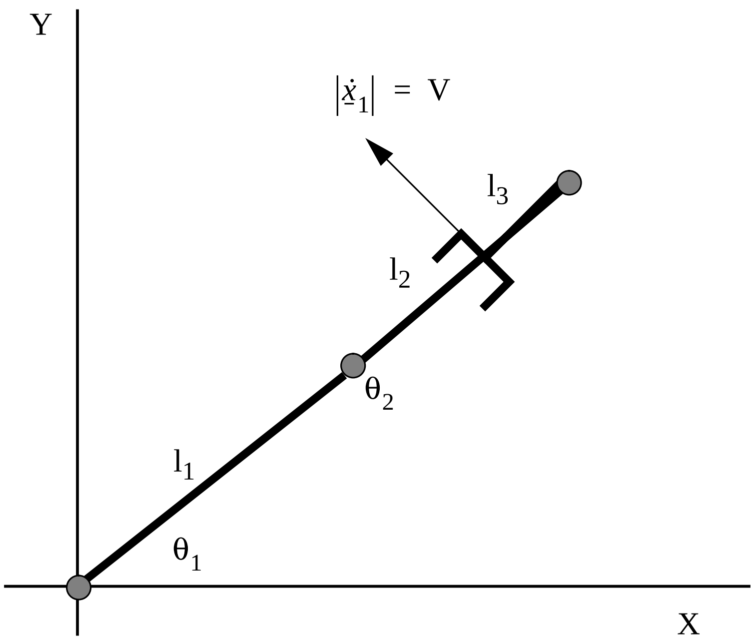
The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.

Velocity at a singularity

Note that although $\dot{\underline{\theta}} = J^{-1}(Q)\,\dot{\underline{x}}$ has no solution in general, if we assume that J loses rank by only one (i.e. r = n-1), then there are n-1 eigenvectors in the task velocity space, $\dot{\underline{x}}_j$, for which solutions *do* exist. However, there will be multiple solutions. For example, if

$$\dot{\underline{x}}_1 = \begin{bmatrix} -Vs_1 \\ Vc_1 \\ 0 \end{bmatrix}$$

The force vector looks like this



We can see (at least geometrically) that the valid solutions for the resulting joint velocities will be

$$\underline{\dot{\theta}}_A = \begin{bmatrix} \dfrac{V}{(l_1 + l_2 - l_3)} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\dot{\theta}}_B = \begin{bmatrix} 0 \\ \dfrac{V}{l_2 - l_3} \\ 0 \end{bmatrix}$$

$$\underline{\dot{\theta}}_C = \begin{bmatrix} 0 \\ 0 \\ \dfrac{V}{-l_3} \end{bmatrix}$$

or, linear combinations of these such as

$$\underline{\dot{\theta}}_n = a_1 \underline{\dot{\theta}}_A + a_2 \underline{\dot{\theta}}_B + a_3 \underline{\dot{\theta}}_C$$

where

$$a_1 + a_2 + a_3 = 1$$

This can be verified by multiplying any of the solutions by $J(\underline{\theta})$ to obtain $\underline{\dot{x}}_1$.

However, if there is any component of tip velocity in the direction corresponding to the zero eigenvalue of J, then at least one joint rate will go to infinity.

Advanced Topic: Coordinate Free Jacobian Matrix

Source: Lecture notes by Dr. Amir Fijany, USC/JPL, 1990

We earlier described the elements of J as the “effective radius” between task axis i and the end effector. Consider the propagation of angular velocity for an all revolute manipulator which we used to compute the bottom three rows of the Jacobian matrix

$$^{n}\omega_{n} = {}^{n}_{n-1}R\left({}^{n-1}\omega_{n-1} + {}^{n-1}Z_{n-1}\dot{\theta}_{n}\right)$$

This is also true *independent of any particular frame*. That is

$$\omega_{n} = \omega_{n-1} + Z_{n-1}\dot{\theta}_{n}$$

as long as all of the vector quantities in this equation, ω_n, ω_{n-1} and Z_{n-1} are ultimately evaluated in the same frame. This is a *coordinate free representation* of the propagation of velocity. Since the coordinate free form is so simple, let’s expand the recursion for a six axis manipulator:

$$\omega_1 = Z_1\dot{\theta}_1 + 0$$

$$\omega_2 = Z_1\dot{\theta}_1 + Z_2\dot{\theta}_2$$

etc. Which gives finally,

$$\omega_6 = \sum_{n=1}^{6} Z_n\dot{\theta}_n$$

which is

$$\omega_6 = \begin{bmatrix} \begin{bmatrix} Z_1 \end{bmatrix} & \begin{bmatrix} Z_2 \end{bmatrix} & \cdots & \begin{bmatrix} Z_6 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \ldots \\ \dot{\theta}_6 \end{bmatrix}$$

Note that the matrix formed by assembling the Z_n vectors into columns is simply the bottom three rows of the Jacobian matrix (also in coordinate free form). By a similar but slightly more complicated derivation:

$$\underline{\dot{x}}_6 = \begin{bmatrix} \begin{bmatrix} Z_1 \otimes P_{1,6} \end{bmatrix} & \begin{bmatrix} Z_2 \otimes P_{2,6} \end{bmatrix} & \dots & \begin{bmatrix} Z_6 \otimes P_{6,6} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dots \\ \dot{\theta}_6 \end{bmatrix}$$

Where $P_{n,6}$ is the vector connecting the origin of frame 6 with the origin of frame n. Note that $P_{6,6}$ is zero and so the last column above should also be zero. This is correct because $\dot{\theta}_6$ should make no contribution to the *linear* velocity of frame 6.

The overall Jacobian in coordinate free form is obtained by combining the two velocity results:

$$\underline{\dot{x}} = J_6 \underline{\dot{\theta}}$$

where

$$J_6 = \begin{bmatrix} \begin{bmatrix} Z_1 \otimes P_{1,6} \end{bmatrix} & \begin{bmatrix} Z_2 \otimes P_{2,6} \end{bmatrix} & \dots & \begin{bmatrix} Z_6 \otimes P_{6,6} \end{bmatrix} \\ \begin{bmatrix} Z_1 \end{bmatrix} & \begin{bmatrix} Z_2 \end{bmatrix} & \dots & \begin{bmatrix} Z_6 \end{bmatrix} \end{bmatrix}$$

The projection of this Jacobian onto any desired frame is “simply” obtained by representing each vector in the desired frame, e.g.

$${}^0J_6 = \begin{bmatrix} \begin{bmatrix} {}^0Z_1 \otimes {}^0P_{1,6} \end{bmatrix} & \begin{bmatrix} {}^0Z_2 \otimes {}^0P_{2,6} \end{bmatrix} & \dots & \begin{bmatrix} {}^0Z_6 \otimes {}^0P_{6,6} \end{bmatrix} \\ \begin{bmatrix} {}^0Z_1 \end{bmatrix} & \begin{bmatrix} {}^0Z_2 \end{bmatrix} & \dots & \begin{bmatrix} {}^0Z_6 \end{bmatrix} \end{bmatrix}$$

and, ${}^{n}J_6$ can be transformed to any other frame by

$$ {}^{m}J_6 = \begin{bmatrix} \left[{}^{m}_{n}R\right] & 0 \\ 0 & \left[{}^{m}_{n}R\right] \end{bmatrix} {}^{n}J_6 $$

where ${}^{m}_{n}R$ is the rotation matrix describing the relative orientation between frames *m* and *n* and 0 is the 3x3 matrix of zero elements. The two R's simply rotate the vectorial elements of the two rows of the Jacobian matrix.

Similarly, we can change the point at which velocity is computed from the origin of frame 6 to any point, r, which is rigidly connected to F_6 by

$$ J_r = \begin{bmatrix} [I] & \left[\hat{P}_{6,r}\right] \\ 0 & [I] \end{bmatrix} J_6 $$

where $[I]$ is the 3x3 identity matrix, and $\hat{P}_{6,r}$ represents the skew-symmetric matrix which encodes the operator $P_{6,r} \otimes [\ ]$. When projected into some frame, *n*,

$$ {}^{n}\hat{P}_{6,r} = \begin{bmatrix} 0 & {}^{n}P_z & -{}^{n}P_y \\ -{}^{n}P_z & 0 & {}^{n}P_x \\ {}^{n}P_y & -{}^{n}P_x & 0 \end{bmatrix} $$